**EE5120 Linear Algebra: Tutorial 7, July-Dec 2017-18**

Covers sec 5.3 (only powers of a matrix part), 5.5,5.6 of GS

1. Prove that the eigenvectors corresponding to different eigenvalues are orthonormal for unitary matrices.

   *Hint:* Use properties of unitary matrices

2. Suppose $A = \begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix}$, where $b$ and $c$ are some non-zero finite real numbers with $c \neq 1$. Compute the eigenvalues and eigenvectors of $A$. Further, if $X = \begin{bmatrix} A & O \\ O & A \end{bmatrix}$, which is a block diagonal matrix and $O$ is an all-zero $2 \times 2$ matrix, then what are its eigenvalues and eigenvectors?

   *Hint:* Use basic procedure for finding eigenvalues and eigenvectors.

3. Let $A$ be an $n \times n$ hermitian matrix with $n$ distinct positive eigen values and for any column vector $\mathbf{x}$, let $\mathbf{x}(k)$ denote the $k^{th}$ entry in $\mathbf{x}$.

   (a) Prove that if $A = U\Lambda U^{-1}$ is the eigen decomposition of $A$, then (i) entries of $\Lambda$ are real, and (ii) $U$ is a unitary matrix.

   *Hint:* First prove that (i) $\mathbf{y}^H\mathbf{A}\mathbf{y}$ is real for any $\mathbf{y}$, (ii) columns of $U$ can be orthonormal.

   (b) Let $\mathbf{u}_2 = \mathbf{u}_1 - aA\mathbf{u}_1$, $\mathbf{v}_i = U^H\mathbf{u}_i$, $i = 1,2$ and $a$ is a real positive number. If $|\mathbf{v}_2(k)| < |\mathbf{v}_1(k)|$, $\forall k$, then prove that $a < \frac{2}{\lambda_{\max}}$, where $\lambda_{\max}$ is the maximum eigen value of $A$.

   *Hint:* Replace $A$ in terms of $\Lambda$ and $U$, and then proceed.

4. Find a third column so that $U$ is unitary. How much freedom in column 3?

$$U = \begin{bmatrix} 1/\sqrt{3} & i/\sqrt{2} & x \\ 1/\sqrt{3} & 0 & y \\ i/\sqrt{3} & 1/\sqrt{2} & z \end{bmatrix}$$

   *Hint:* Use the definition of unitary matrix.

5. Suppose each "Gibonacci" number $G_{k+2}$ is the average of the two previous numbers $G_{k+1}$ and $G_k$. Then $G_{k+2} = \frac{1}{2}(G_{k+1} + G_k)$ : $\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$

   (a) Find the eigenvalues and eigenvectors of $A$.

   *Hint:* Use basic procedure for finding eigenvalues and eigenvectors.

   (b) Find the limit as $n \to \infty$ of the matrices $A^n = S\Lambda^n S^{-1}$.

   *Hint:* Compute $\Lambda^\infty$.

   (c) If $G_0 = 0$ and $G_1 = 1$, show that the Gibonacci numbers approach $\frac{2}{3}$.

   *Hint:* Compute $A^\infty \begin{bmatrix} G_1 \\ G_0 \end{bmatrix}$.

6. Suppose there is an epidemic in which every month half of those who are well become sick, and a quarter of those who are sick become dead. Find the steady state for the corresponding Markov process:

$$\begin{bmatrix} d_{k+1} \\ s_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} d_k \\ s_k \\ w_k \end{bmatrix}$$

*Hint:* Steady state is the eigen vector corresponding eigen value 1.

7. Show that every number is an eigenvalue for the transformation $T(f(x)) = \frac{df}{dx}$, but the transformation $T(f(x)) = \int_0^x f(t)dt$ has no eigenvalues (here $-\infty < x < \infty$).

   *Hint:* Take $f(x) = e^{ax}$ for any real number $a$ and solve.

8. Let $\mathbf{A}$ be a square matrix. Define $\mathbf{B_1}=\frac{1}{2}(\mathbf{A} + \mathbf{A}^H)$, $\mathbf{B_2}=\frac{1}{2i}(\mathbf{A} - \mathbf{A}^H)$
   (a) Prove that $\mathbf{B_1}$, $\mathbf{B_2}$ are hermitian and $\mathbf{A}=\mathbf{B_1}+i\mathbf{B_2}$
   (b) Suppose that $\mathbf{A} = \mathbf{C_1}+i\mathbf{C_2}$, where $\mathbf{C_1}$ and $\mathbf{C_2}$ are hermitian, then prove that $\mathbf{C_1}=\mathbf{B_1}$ and $\mathbf{C_2}=\mathbf{B_2}$
   (c) What conditions of $\mathbf{B_1}$ and $\mathbf{B_2}$ make $\mathbf{A}$ normal ?

   *Hint:* Use definitions.

9. Let $\mathbf{A}$ be a normal matrix. Prove the following:
   (a) $\|\mathbf{Ax}\| = \|\mathbf{A^H x}\|$ for every $\mathbf{x} \in \mathbb{C}^n$
   (b) $\mathbf{A} - c\mathbf{I}$ is a normal operator for every $c \in \mathbb{C}$
   (c) If $\mathbf{x}$ is an eigen vector of $\mathbf{A}$ with eigen value $\lambda$, then $\mathbf{x}$ is also an eigen vector of $\mathbf{A}^H$ with eigen value $\lambda^*$ ($\lambda^*$ is the complex conjugate of $\lambda^*$)

   *Hint:* Use part (a) and part (b) to solve part (c)

10. If the transformation $T$ is a reflection across the 45° line in the plane, find its matrix with respect to the standard basis $v_1 = (1, 0)$, $v_2 = (0, 1)$, and also with respect to $V_1 = (1, 1)$, $V_2 = (1, -1)$. Show that those matrices are similar.

    *Hint:* Use Change of basis or Similarity transformation $B = M^{-1}AM$.