EE5120 Linear Algebra: Tutorial 7, July-Dec 2017-18
Covers sec 5.3 (only powers of a matrix part), 5.5,5.6 of GS

1. Prove that the eigenvectors corresponding to different eigenvalues are orthonormal for unitary matrices.

   Hint: Use properties of unitary matrices

   > **Solution:** See pg. 287 of GS

2. Suppose $A = \begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix}$, where b and c are some non-zero finite real numbers with $c \neq 1$. Compute the eigenvalues and eigenvectors of A. Further, if $X = \begin{bmatrix} A & O \\ O & A \end{bmatrix}$, which is a block diagonal matrix and O is an all-zero 2×2 matrix, then what are its eigenvalues and eigenvectors?

   Hint: Use basic procedure for finding eigenvalues and eigenvectors.

   > **Solution:**
   > A is upper triangular. So, diagonal entries are its eigenvalues, i.e., $\lambda_A = 1, c$. Eigenvectors can be obtained as, $[1\ 0]^T$ and $[\frac{b}{c-1}\ 1]^T$ respectively.
   >
   > X is block diagonal. Since A is upper triangular, so is X. Thus eigen values of X are: $\lambda_X = 1, 1, c, c$ and corresponding eigenvectors are $[1\ 0\ 0\ 0]^T, [0\ 0\ 1\ 0]^T, [\frac{b}{c-1}\ 1\ 0\ 0]^T$ and $[0\ 0\ \frac{b}{c-1}\ 1]^T$ respectively.

3. Let A be an $n \times n$ hermitian matrix with n distinct positive eigen values and for any column vector $\mathbf{x}$, let $\mathbf{x}(k)$ denote the k^{th} entry in $\mathbf{x}$.

   (a) Prove that if $A = U\Lambda U^{-1}$ is the eigen decomposition of A, then (i) entries of Λ are real, and (ii) U is a unitary matrix.

   Hint: First prove that (i) $\mathbf{y}^H\mathbf{A}\mathbf{y}$ is real for any $\mathbf{y}$, (ii) columns of U can be orthonormal.

   (b) Let $\mathbf{u}_2 = \mathbf{u}_1 - aA\mathbf{u}_1$, $\mathbf{v}_i = U^H\mathbf{u}_i$, $i = 1, 2$ and a is a real positive number. If $|\mathbf{v}_2(k)| < |\mathbf{v}_1(k)|$, $\forall k$, then prove that $a < \frac{2}{\lambda_{\max}}$, where $\lambda_{\max}$ is the maximum eigen value of A.

   Hint: Replace A in terms of Λ and U, and then proceed.

   > **Solution:**
   >
   > (a) For any non-zero vector $\mathbf{y}$ of appropriate dimension, note that $\mathbf{y}^H A\mathbf{y}$ is a scalar. Now, $(\mathbf{y}^H A\mathbf{y})^* = (\mathbf{y}^H)^* A^* \mathbf{y}^* = (\mathbf{y}^T A^* \mathbf{y}^*)^T = \mathbf{y}^H A^H \mathbf{y} = \mathbf{y}^H A\mathbf{y}$, where the third equality is due to the fact that $\mathbf{y}^H A\mathbf{y}$ is a scalar. Since $(\mathbf{y}^H A\mathbf{y})^* = \mathbf{y}^H A\mathbf{y}$, $\mathbf{y}^H A\mathbf{y}$ is a real number.
   > Now, let $\mathbf{p}$ be a eigenvector corresponding to eigenvalue λ for A. Then, $\mathbf{p}^H\mathbf{A}\mathbf{p}$ is real by above arguement. And, $\mathbf{p}^H\mathbf{A}\mathbf{p} = \mathbf{p}^H\lambda\mathbf{p} = \lambda||\mathbf{p}||^2$. When LHS is real and $||\mathbf{p}||^2$ is real, λ must also be real. Hence, all eigen values of a hermitian matrix

are real.
Further, let λ_1 and λ_2 be two eigen values of A with eigenvectors $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively. Now, since λ_1 and λ_2 are real and distinct, we have,

$$\lambda_1 \mathbf{p}_1^H \mathbf{p}_2 = (\lambda_1 \mathbf{p}_1)^H \mathbf{p}_2 = (A\mathbf{p}_1)^H \mathbf{p}_2 = \mathbf{p}_1^H A^H \mathbf{p}_2 = \mathbf{p}_1^H A \mathbf{p}_2 = \lambda_2 \mathbf{p}_1^H \mathbf{p}_2.$$

Since eigenvalues are distinct the above relation holds only if $\mathbf{p}_1^H \mathbf{p}_2 \Rightarrow \frac{\mathbf{p}_1^H}{||\mathbf{p}_1||^2} \frac{\mathbf{p}_2}{||\mathbf{p}_2||^2} = 0$. Thus, eigen vectors are orthonormal, which implies U is unitary.

(b) Now,

$$\mathbf{u}_2 = \mathbf{u}_1 - aA\mathbf{u}_1 = \mathbf{u}_1 - aU\Lambda U^H \mathbf{u}_1 = (UU^H - aU\Lambda U^H)\mathbf{u}_1 = U(I - a\Lambda)U^H \mathbf{u}_1.$$

Thus, $U^H \mathbf{u}_2 = (I - a\Lambda)U^H \mathbf{u}_2 \Rightarrow \mathbf{v}_2 = (I - a\Lambda)\mathbf{v}_1$. Let k^{th} diagonal entry in Λ be λ_k. Then, $|\mathbf{v}_2(k)| = |1 - a\lambda_k||\mathbf{v}_1(k)|$, $\forall k$. If we want $|\mathbf{v}_2(k)| < |\mathbf{v}_1(k)|$, $\forall k$, then we must have $|1 - a\lambda_k| < 1$, $\forall k \Rightarrow -1 < 1 - a\lambda_k < 1 \Rightarrow 0 < a < \frac{2}{\lambda_k}$. Since, this is true for all k, we have $0 < a < \frac{2}{\lambda_{\max}}$, where $\lambda_{\max}$ is the maximum eighen value of A.

4. Find a third column so that U is unitary. How much freedom in column 3?

$$U = \begin{bmatrix} 1/\sqrt{3} & i/\sqrt{2} & x \\ 1/\sqrt{3} & 0 & y \\ i/\sqrt{3} & 1/\sqrt{2} & z \end{bmatrix}$$

Hint: Use the definition of unitary matrix.

Solution: Given $U^H U = I$. On equating LHS and RHS, we get 9 equations of which 4 are redundant. The other 5 equations are given below ,

$$\frac{1}{3} + \frac{1}{2} + x\bar{x} = 1 \quad (1)$$

$$\frac{1}{3} + y\bar{y} = 1 \quad (2)$$

$$\frac{1}{3} + \frac{1}{2} + z\bar{z} = 1 \quad (3)$$

$$\frac{1}{3} + x\bar{y} = 0 \quad (4)$$

$$\frac{-i}{3} + \frac{i}{2} + x\bar{z} = 0 \quad (5)$$

From equations (1), (2) and (3),
$x = \frac{1}{\sqrt{6}} \exp{(i\theta_x)}$, $y = \sqrt{\frac{2}{3}} \exp{(i\theta_y)}$ and $z = \frac{1}{\sqrt{6}} \exp{(i\theta_z)}$, respectively.
From euqations (4) and (5),
$\theta_x - \theta_y = (2n+1)\pi$ and $\theta_x - \theta_z = (2k - \frac{1}{2})\pi$, respectively. So, $x = \frac{1}{\sqrt{6}} \exp{(i\theta_x)}$,
$y = \sqrt{\frac{2}{3}} \exp i(\theta_x - (2n+1)\pi) = \sqrt{\frac{2}{3}} \exp i(\theta_x - \pi)$
and $z = \frac{1}{\sqrt{6}} \exp i(\theta_x - (2k - \frac{1}{2})\pi) = \frac{1}{\sqrt{6}} \exp i(\theta_x + \frac{\pi}{2})$.

The third column is "$\exp(i\theta_x) \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \exp(-i\pi) \\ \frac{1}{\sqrt{6}} \exp(i\frac{\pi}{2}) \end{bmatrix}$", which has no degree of freedom.

5. Suppose each "Gibonacci" number G_{k+2} is the average of the two previous numbers G_{k+1} and G_k. Then $G_{k+2} = \frac{1}{2}(G_{k+1} + G_k) : \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$

   (a) Find the eigenvalues and eigenvectors of A.

   Hint: Use basic procedure for finding eigenvalues and eigenvectors.

   (b) Find the limit as $n \to \infty$ of the matrices $A^n = S\Lambda^n S^{-1}$.

   Hint: Compute Λ^∞.

   (c) If $G_0 = 0$ and $G_1 = 1$, show that the Gibonacci numbers approach $\frac{2}{3}$.

   Hint: Compute $A^\infty \begin{bmatrix} G_1 \\ G_0 \end{bmatrix}$.

Solution:

(a) $A = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}$. Eigenvalues ($\lambda$, s.t. $det(A - \lambda I) = 0$) are 1 and -0.5, with eigenvectors (v_i, s.t. $Av_i = \lambda v_i$) are $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

(b) $S = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{bmatrix}$, $S^{-1} = \frac{\sqrt{10}}{3} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

$A^\infty = S \begin{bmatrix} 1 & 0 \\ 0 & -0.5^\infty \end{bmatrix} S^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

(c) $\begin{bmatrix} G_\infty \\ G_{\infty-1} \end{bmatrix} = A^\infty \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} = A^\infty \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$

6. Suppose there is an epidemic in which every month half of those who are well become sick, and a quarter of those who are sick become dead. Find the steady state for the corresponding Markov process:

$$\begin{bmatrix} d_{k+1} \\ s_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} d_k \\ s_k \\ w_k \end{bmatrix}$$

Hint: Steady state is the eigen vector corresponding eigen value 1.

Solution: Let us denote the given matrix equation as $\mathbf{u_{k+1}} = A\mathbf{u_k}$ where $\mathbf{u_{k+1}} = \begin{bmatrix} d_{k+1} \\ s_{k+1} \\ w_{k+1} \end{bmatrix}$, $A = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ and $\mathbf{u_k} = \begin{bmatrix} d_k \\ s_k \\ w_k \end{bmatrix}$.

The seady state is given by $A\mathbf{u}_\infty = \mathbf{u}_\infty$, i.e., $\mathbf{u}_\infty$ is the eigenvector corresponding to eigenvalue 1 (A Markov matrix will always have one of the eigenvalues as 1 and the rest of magnitude ≤ 1). To find it, we equate $(A - 1I)\mathbf{u}_\infty = \mathbf{0}$.

$$\begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 0 & \frac{-1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} d_\infty \\ s_\infty \\ w_\infty \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives $\mathbf{u}_\infty = \begin{bmatrix} d_\infty \\ s_\infty \\ w_\infty \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, in other words, everyone will die from the epidemic.

7. Show that every number is an eigenvalue for the transformation $T(f(x)) = \frac{df}{dx}$, but the transformation $T(f(x)) = \int_0^x f(t)dt$ has no eigenvalues (here $-\infty < x < \infty$).

Hint: Take $f(x) = e^{ax}$ for any real number a and solve.

Solution: If $f(x) = e^{ax}$ for any real number a, for the first T,

$$T(f(x)) = \frac{df}{dx} = ae^{ax} = af(x)$$

Hence, any number a is an eigenvalue of T. [Note: the eigenvalues of a linear transformation are not dependent on a basis, so if we have found them in one basis that is good enough. Why? Any change of basis is given by a similarity transform, and it is easy to prove that similar matrices have the same eigenvalues. In this case we chose $f(x) = e^{ax}$ because it was an easy to guess eigenfunction. Since the question is regarding eigenvalues, it is important to only choose those f that are eigenfunctions, not arbitrary functions.]

For the second T, suppose $T(f(x)) = af(x)$ for some number a and some function f, i.e.,

$$\int_0^x f(t)dt = af(x)$$

Now, differentiate both sides to get

$$f(x) = af'(x)$$

Solving for f, we get

$$\int \frac{f'(x)dx}{f(x)} = \int \frac{1}{a}dx$$

This gives

$$ln|f(x)| = \frac{x}{a} + C \quad \text{or} \quad |f(x)| = e^{\frac{x}{a}+C} \quad \text{or} \quad |f(x)| = e^C e^{\frac{x}{a}}$$

We can get rid of the absolute value signs by substituting A for e^C (allowing A to possibly be negative):

$$f(x) = Ae^{\frac{x}{a}}$$

Therefore we have

$T(f(x)) = \int_0^x f(t)dt = \int_0^x Ae^{\frac{t}{a}}dt = aAe^{\frac{t}{a}}|_o^x = aAe^{\frac{x}{a}} - aA = a(Ae^{\frac{x}{a}} - A) = a(f(x) - A)$

But our initial assumption was that $T(f(x) = af(x)$. For this either $a = 0$ or $A = 0$, either of which implies $f(x) = 0$. Hence T has no eigenvalues.

8. Let $\mathbf{A}$ be a square matrix. Define $\mathbf{B_1}$=$\frac{1}{2}(\mathbf{A} + \mathbf{A}^H)$, $\mathbf{B_2}$=$\frac{1}{2i}(\mathbf{A} - \mathbf{A}^H)$
   (a) Prove that $\mathbf{B_1}$, $\mathbf{B_2}$ are hermitian and $\mathbf{A}$=$\mathbf{B_1}$+$i\mathbf{B_2}$
   (b) Suppose that $\mathbf{A} = \mathbf{C_1}$+$i\mathbf{C_2}$, where $\mathbf{C_1}$ and $\mathbf{C_2}$ are hermitian, then prove that $\mathbf{C_1}$=$\mathbf{B_1}$ and $\mathbf{C_2}$=$\mathbf{B_2}$
   (c) What conditions of $\mathbf{B_1}$ and $\mathbf{B_2}$ make $\mathbf{A}$ normal ?

Hint: Use definitions.

Solution:

(a) $\mathbf{B_1}^H = \frac{1}{2}(\mathbf{A}^H + \mathbf{A}) = \mathbf{B_1}$ similarly, $\mathbf{B_2}^H = \frac{-1}{2i}(\mathbf{A}^H - \mathbf{A}) = \mathbf{B_2}$ and $\mathbf{B_1}$+i$\mathbf{B_2}$=$\frac{1}{2}(\mathbf{A} + \mathbf{A}^H)$+$\frac{1}{2}(\mathbf{A} - \mathbf{A}^H) = \mathbf{A}$

(b) $\mathbf{A} = \mathbf{C_1}$+$i\mathbf{C_2}$,$\Rightarrow$ $\mathbf{A}^H = \mathbf{C_1}^H$+$i\mathbf{C_2}^H = \mathbf{C_1}$- $i\mathbf{C_2}$ using the exressions of $\mathbf{A}$ and $\mathbf{A}^H$, we get $\mathbf{C_1}$=$\frac{1}{2}(\mathbf{A} + \mathbf{A}^H) = \mathbf{B_1}$ and $\mathbf{C_2}$=$\frac{1}{2i}(\mathbf{A} - \mathbf{A}^H) = \mathbf{B_2}$

(c) $\mathbf{A}\mathbf{A}^H$=$\mathbf{A}^H\mathbf{A}$ $\Rightarrow$ ($\mathbf{B_1}$+$i\mathbf{B_2}$)($\mathbf{B_1}$-$i\mathbf{B_2}$)=($\mathbf{B_1}$-$i\mathbf{B_2}$)($\mathbf{B_1}$+$i\mathbf{B_2}$) simplification yields $\mathbf{B_1}\mathbf{B_2}$=$\mathbf{B_2}\mathbf{B_1}$

9. Let $\mathbf{A}$ be a normal matrix. Prove the following:
   (a) $\|\mathbf{A}\mathbf{x}\| = \|\mathbf{A^H}\mathbf{x}\|$ for every $\mathbf{x} \in \mathbb{C}^n$
   (b) $\mathbf{A} - c\mathbf{I}$ is a normal operator for every $c \in \mathbb{C}$
   (c) If $\mathbf{x}$ is an eigen vector of $\mathbf{A}$ with eigen value λ, then $\mathbf{x}$ is also an eigen vector of $\mathbf{A}^H$ with eigen value λ^* (λ^* is the complex conjugate of λ^*)

Hint: Use part (a) and part (b) to solve part (c)

Solution:

(a) $\|\mathbf{A}\mathbf{x}\|^2$=$(\mathbf{A}\mathbf{x})^H\mathbf{A}\mathbf{x} = \mathbf{x}^H\mathbf{A}^H\mathbf{A}\mathbf{x} = \mathbf{x}^H\mathbf{A}\mathbf{A}^H\mathbf{x} = (\mathbf{A^H}\mathbf{x})^H\mathbf{A^H}\mathbf{x} = \|\mathbf{A^H}\mathbf{x}\|^2$

(b) $(\mathbf{A} - c\mathbf{I})(\mathbf{A} - c\mathbf{I})^H = \mathbf{A}\mathbf{A}^H - c^*\mathbf{A} - c\mathbf{A}^H + |c|^2\mathbf{I} = \mathbf{A}^H\mathbf{A} - c^*\mathbf{A} - c\mathbf{A}^H + |c|^2\mathbf{I} = (\mathbf{A} - c\mathbf{I})^H(\mathbf{A} - c\mathbf{I})$ therefore normal

(c) Let $\mathbf{x}$ be an eigen vector of $\mathbf{A}$ with eigen value $\lambda \Rightarrow \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Define $\mathbf{U} = \mathbf{A} - \lambda\mathbf{I}$. From part b, $\mathbf{U}$ is normal. From part a, $\|\mathbf{U}\mathbf{x}\| = \|\mathbf{U^H}\mathbf{x}\| \Rightarrow 0 = \|(\mathbf{A} - \lambda\mathbf{I})\mathbf{x}\| = \|\mathbf{A}^H\mathbf{x} - \lambda^*\mathbf{x}\| \Rightarrow \mathbf{A}^H\mathbf{x} = \lambda^*\mathbf{x}$

10. If the transformation T is a reflection across the 45° line in the plane, find its matrix with respect to the standard basis $v_1 = (1,0)$, $v_2 = (0,1)$, and also with respect to $V_1 = (1,1)$, $V_2 = (1,-1)$. Show that those matrices are similar.

Hint: Use Change of basis or Similarity transformation $B = M^{-1}AM$.

Solution: Let T be the Transformation matrix for reflection across any θ line given by,

$$T = \begin{bmatrix} 2c^2 - 1 & 2cs \\ 2cs & 2s^2 - 1 \end{bmatrix}$$

Here, $\theta = 45^\circ$. Substituting this value, we get the transformation matrix wrt standard basis v_1, v_2. Lets call this matrix A, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Now, we change the basis to $V_1 = (1,1), V_2 = (1,-1)$ and find the reflection transformation matrix (call it B). Observe where the new basis vectors land after reflection through the given line,

$$TV_1 = V_1$$

$$TV_2 = -V_2$$

Thus, the transformation matrix wrt V_1, V_2 will be $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Next, to show that A and B are similar matrices, we need to find a matrix M such that, $B = M^{-1}AM$. Matrix M is constructed by writing new basis vectors V_1, V_2 as a linear combination of standard basis vectors v_1, v_2. Since $(1,1) = 1(1,0) + 1(0,1)$ and $(1,-1) = 1(1,0) - 1(0,1)$, we get

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Using this M, it can be verified that $B = M^{-1}AM$. Note that, $(1,1)$ and $(1,-1)$ are also the eigenvectors of A (or T).