EE5120 Linear Algebra: Tutorial 5, July-Dec 2017-18

1. (a) Suppose $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a set of non-zero orthogonal vectors, then prove that $\mathcal{S}$ is linearly independent.

   Hint: Prove by contradiction.

   (b) Let V be a finite dimensional vector space with $\{\mathbf{u}_1, ..., \mathbf{u}_k\}$ being its orthonormal basis vectors. If for some $\mathbf{v} \in \mathsf{V}$, we have $\mathbf{v} = \sum_{i=1}^{k} a_i \mathbf{u}_i$, then find an expression for the scalars a_i's in terms of$\mathbf{u}_i$'s and $\mathbf{v}$.

   Hint: Multiply $\mathbf{v}^T$ with $\mathbf{u}_i$'s & observe.

   (c) Using Gram-Schmidt procedure find orthonormal vectors $\mathbf{q}_1, \mathbf{q}_2$ and $\mathbf{q}_3$ corresponding to the vectors $[1\,1\,0]^T$, $[1\,0\,1]^T$ and $[0\,1\,1]^T$.

   Hint: Follow the procedure.

   (d) Suppose $\mathcal{S} = \{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n\}$ is a set of $n \times 1$ orthonormal vectors. Prove that Gram-Schmidt procedure when applied to $\mathcal{S}$ leads to $\mathcal{S}$ itself. Further if $\mathbf{a}_i$ is the i^{th} column of matrix $\mathbf{A}$, then show that $\mathbf{R}$ matrix in the QR-factorization of $\mathbf{A}$ is an identity matrix.

   Hint: Mathematical induction.

2. (a) Suppose V is a vector space and W is a sub-space. If $\mathsf{U} = \{\mathbf{v} \in \mathsf{V} | \mathbf{v}^T\mathbf{w} = 0, \forall \mathbf{w} \in \mathsf{W}\}$, prove that U is a sub-space.

   (b) Let $\mathbf{u} \in \mathbb{R}^n$ is s.t. $\mathbf{u}^T\mathbf{u} = 1$ and $\mathbf{Q} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$, where $\mathbf{I}$ is an $n \times n$ identity matrix.

   (i) Prove that $\mathbf{Q}$ is an orthogonal matrix.

   (ii) If $\mathbf{w} \in \mathbb{R}^n$ is s.t. $\mathbf{Q}\mathbf{w} = \mathbf{w}$, compute the relation between $\mathbf{w}$ and $\mathbf{u}$.

   Hint: Just follow the actual definitions.

3. (a) Square the matrix $P = aa^T/(a^Ta)$, which projects onto a line, and show that $P^2 = P$.

   Hint: Note the number a^Ta in the middle of the matrix aa^Taa^T!

   (b) Find the matrix P that projects every vector b in R^3 onto the line in the direction of a and the projection p. Check that error $e = b - p$ is perpendicular to a for the following:

   (i) $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $a = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ (ii) $b = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$ and $a = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$

   Hint: $p = Pb$ and show that $e^Ta = 0$.

   (c) Is the projection matrix P invertible? Why or why not?

   Hint: Use the rank of P matrix to support your statement.

4. A Middle-Aged man was stretched on a rack to lengths $L = 5, 6,$ and 7 feet under applied forces of $F = 1, 2,$ and 4 tons. Assuming Hooke's law $L = a + bF$, find his normal length a by least squares.

   Hint: Least square solution of $Ax = y$ is $\hat{x} = (A^TA)^{-1}A^Ty$. Find A, x, and b in the problem.

5. Let $\mathbf{W} = span(\{(i, 0, 1)\})$ in complex-valued vector space of dimension 3. Find orthonormal bases for $\mathbf{W}$ and $\mathbf{W}^\perp$.

   Hint: Treat complex-valued vector as any other vector and apply orthogonality principle.

6. Find the parabola $C + Dt + Et^2$ that comes closest to the values $\mathbf{b} = (0, 0, 1, 0, 0)$ at the times $t = -2, -1, 0, 1, 2$.

   Hint: Here, best parabola is symmetric about time t. This directly tells about value of D.

7. If $\mathbf{T}$ is a linear transformation on a vector space $\mathbf{V}$ such that $\|\mathbf{T}(\mathbf{x})\| = \|\mathbf{x}\|$ for $\mathbf{x} \in \mathbf{V}$, prove that it is one to one

   Hint: What happens when $\mathbf{x} \in \mathbf{N}(\mathbf{T})$, *where* $\mathbf{N}(\mathbf{T})$ *is the null space of* $\mathbf{T}$?

8. Let $\mathbf{V}$ be an innear product space and define for each pair of vectors $\mathbf{x}, \mathbf{y}$, the scalar $\mathbf{d}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$, called distance between $\mathbf{x}$ and $\mathbf{y}$. Prove for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ that

   (a) $\mathbf{d}(\mathbf{x}, \mathbf{y}) \geq 0$
   (b) $\mathbf{d}(\mathbf{x}, \mathbf{y}) = \mathbf{d}(\mathbf{y}, \mathbf{x})$
   (c) $\mathbf{d}(\mathbf{x}, \mathbf{y}) \leq \mathbf{d}(\mathbf{x}, \mathbf{z}) + \mathbf{d}(\mathbf{z}, \mathbf{y})$
   (d) $\mathbf{d}(\mathbf{x}, \mathbf{x})$=0
   (e) $\mathbf{d}(\mathbf{x}, \mathbf{y}) \neq 0$ if $\mathbf{x} \neq \mathbf{y}$

   Hint: Use the definitions

9. The fundamental theorem is often stated in the form of *Fredholms alternative*: For any A and $\mathbf{b}$, one and only one of the following systems has a solution:
   (i) $A\mathbf{x} = \mathbf{b}$ (or)
   (ii) $A^T\mathbf{y} = \mathbf{0}$ with $\mathbf{y}^T\mathbf{b} \neq 0$.
   Either $\mathbf{b}$ is in the column space $C(A)$ or there is a $\mathbf{y}$ in $N(A^T)$ such that $\mathbf{y}^T\mathbf{b} = 0$. Show that it is contradictory for (i) and (ii) both to have solutions.

   Hint: Proof by contradiction. Start by expressing 0 as $\mathbf{0}^T\mathbf{x}$.

10. Why is each of these statements false?

    (a) (1,1,1) is perpendicular to (1,1,-2), so the planes $x + y + z = 0$ and $x + y - 2z = 0$ are orthogonal subspaces.

    Hint: Check dimensions of the subspaces.

    (b) The subspace spanned by (1,1,0,0,0) and (0,0,0,1,1) is the orthogonal complement of the subspace spanned by (1,-1,0,0,0) and (2,-2,3,4,-4).

    Hint: Check dimensions of the subspaces.

    (c) Two subspaces that meet only in the zero vector are orthogonal.

    Hint: Take a simple example in 2D space to disprove.