EE5120 Linear Algebra: Tutorial 5, July-Dec 2017-18

1. (a) Suppose $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a set of non-zero orthogonal vectors, then prove that $\mathcal{S}$ is linearly independent.

   Hint: Prove by contradiction.

   (b) Let V be a finite dimensional vector space with $\{\mathbf{u}_1, ..., \mathbf{u}_k\}$ being its orthonormal basis vectors. If for some $\mathbf{v} \in \mathsf{V}$, we have $\mathbf{v} = \sum_{i=1}^{k} a_i \mathbf{u}_i$, then find an expression for the scalars a_i's in terms of$\mathbf{u}_i$'s and $\mathbf{v}$.

   Hint: Multiply $\mathbf{v}^T$ with $\mathbf{u}_i$'s & observe.

   (c) Using Gram-Schmidt procedure find orthonormal vectors $\mathbf{q}_1, \mathbf{q}_2$ and $\mathbf{q}_3$ corresponding to the vectors $[1\,1\,0]^T$, $[1\,0\,1]^T$ and $[0\,1\,1]^T$.

   Hint: Follow the procedure.

   (d) Suppose $\mathcal{S} = \{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n\}$ is a set of $n \times 1$ orthonormal vectors. Prove that Gram-Schmidt procedure when applied to $\mathcal{S}$ leads to $\mathcal{S}$ itself. Further if $\mathbf{a}_i$ is the i^{th} column of matrix $\mathbf{A}$, then show that $\mathbf{R}$ matrix in the QR-factorization of $\mathbf{A}$ is an identity matrix.

   Hint: Mathematical induction.

Solution:

(a) Assume that $\mathcal{S}$ is linearly dependent. Then there exists scalars $a_1, ..., a_n$ s.t. $\sum_{i=1}^{n} a_i \mathbf{v}_i = \mathbf{0}$, with not all scalars being equal to zero. Now, $\mathbf{v}_1^T(\sum_{i=1}^{n} a_i \mathbf{v}_i) = \mathbf{v}_1^T \mathbf{0} \Rightarrow a_1 \mathbf{v}_1^T \mathbf{v}_1 = \mathbf{0}$. The above equation is due to orthogonality between $\mathbf{v}_i$'s. Since $\mathbf{v}_i$'s are non-zero $\mathbf{v}_1^T \mathbf{v}_1 > 0 \Rightarrow a_1 = 0$. By this procedure, we can obtain that $a_i = 0$, $\forall i = 1, 2, ..., n$. Arrived at a contradiction. Thus, our assumption is incorrect $\Rightarrow \mathcal{S}$ is linearly independent.

(b) For some $i = 1, ..., k$, $\mathbf{v}^T \mathbf{u}_i = \sum_{j=1}^{k} a_j \mathbf{u}_j^T \mathbf{u}_i = a_i \mathbf{u}_i^T \mathbf{u}_i = a_i.1 = a_i$. Thus, $a_i = \mathbf{v}^T \mathbf{u}_i$, $\forall i = 1, ..., n$.

(c) The orthonormal vectors are: $[\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\ 0]^T$, $[\frac{1}{\sqrt{2}}\ -\frac{1}{\sqrt{2}}\ 0]^T$ and $[0\,0\,1]^T$.

(d) Let $\{\mathbf{q}_i\}_{i=1}^{n}$ be the result of Gram-Schmidt procedure. We prove the result by mathematical induction on the i. For $i = 1$, $\mathbf{q}_1$ is equal to $\mathbf{a}_1$ since $\mathbf{a}_1$ is s.t. $\mathbf{a}_1^T \mathbf{a}_1 = 1$. Hence the result holds true for $i = 1$. Let it hold true for first $i - 1$ vectors, i.e., $\mathbf{q}_j = \mathbf{a}_j$, $\forall j = 1, ..., i-1$. In the i^{th} step of Gram-Schmidt algorithm, $\mathbf{q}_i = \mathbf{a}_i - \sum_{k=1}^{i-1} \mathbf{a}_i^T \mathbf{q}_k \mathbf{q}_k = \mathbf{a}_i - \sum_{k=1}^{i-1} (\mathbf{a}_i^T \mathbf{a}_k) \mathbf{a}_k = \mathbf{a}_i - \sum_{k=1}^{i-1} (0) \mathbf{a}_k = \mathbf{a}_i$ [By orthogonality of $\mathbf{a}_j$'s]. Thus, it is true for all $i = 1, ..., n$. Further, $\mathbf{R}$ matrix in QR-factorization of $\mathbf{A} = [\mathbf{a}_1 \dots \mathbf{a}_n]$ is an upper triangular matrix whose non-zero entries are $[\mathbf{R}]_{ij} = \mathbf{q}_i^T \mathbf{a}_j$, $i, j = 1, ..., n$ and $j \geq i$. Hence, $[\mathbf{R}]_{ij} = \mathbf{a}_i^T \mathbf{a}_j = 1$, if $i = j$ and zero if $i \neq j$. Thus, $\mathbf{R}$ is an $n \times n$ identity matrix.

2. (a) Suppose V is a vector space and W is a sub-space. If $\mathsf{U} = \{\mathbf{v} \in \mathsf{V} | \mathbf{v}^T \mathbf{w} = 0, \forall \mathbf{w} \in \mathsf{W}\}$, prove that U is a sub-space.

   (b) Let $\mathbf{u} \in \mathbb{R}^n$ is s.t. $\mathbf{u}^T \mathbf{u} = 1$ and $\mathbf{Q} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$, where $\mathbf{I}$ is an $n \times n$ identity matrix.

      (i) Prove that $\mathbf{Q}$ is an orthogonal matrix.

(ii) If $\mathbf{w} \in \mathbb{R}^n$ is s.t. $\mathbf{Qw} = \mathbf{w}$, compute the relation between $\mathbf{w}$ and $\mathbf{u}$.

Hint: Just follow the actual definitions.

Solution:

(a) Since $\mathbf{0}^T\mathbf{w} = 0$, $\forall \mathbf{w} \in \mathrm{W}$, $\mathbf{0} \in \mathrm{U}$. Suppose $\mathbf{a}, \mathbf{b} \in \mathrm{U}$, then $\mathbf{a}^T\mathbf{w} = 0 = \mathbf{b}^T\mathbf{w}$, $\forall \mathbf{w} \in \mathrm{W}$. Now, $(\mathbf{a}+\mathbf{b})^T\mathbf{w} = (\mathbf{a}^T\mathbf{w} + \mathbf{b}^T\mathbf{w}) = 0 + 0 = 0 \Rightarrow \mathbf{a}+\mathbf{b} \in \mathrm{U}$. Lastly, if $\mathbf{a} \in \mathrm{U} \Rightarrow \mathbf{a}^T\mathbf{w} = 0$, $\forall \mathbf{w} \in \mathrm{W}$, then for any scalar c, $(c\mathbf{a})^T\mathbf{w} = c^T\mathbf{a}^T\mathbf{w} = c.0 = 0 \Rightarrow c\mathbf{a} \in \mathrm{U}$. Hence, U is a sub-space.

(b) (i) Note that $\mathbf{Q}^T = (\mathbf{I} - 2\mathbf{uu}^T)^T = \mathbf{I}^T - 2(\mathbf{u}^T)^T\mathbf{u}^T = \mathbf{Q}$. Thus, $\mathbf{QQ}^T = \mathbf{Q}^T\mathbf{Q} = (\mathbf{I} - 2\mathbf{uu}^T)(\mathbf{I} - 2\mathbf{uu}^T) = \mathbf{I} - 2\mathbf{uu}^T - 2\mathbf{uu}^T + 4\mathbf{uu}^T\mathbf{uu}^T = \mathbf{I} - 4\mathbf{uu}^T + 4\mathbf{u}(\mathbf{u}^T\mathbf{u})\mathbf{u}^T = \mathbf{I} - 4\mathbf{uu}^T + 4\mathbf{u}(1)\mathbf{u}^T = \mathbf{I}$.

(b) $\mathbf{Qw} = \mathbf{w} - 2\mathbf{uu}^T\mathbf{w} = \mathbf{w} \Rightarrow \mathbf{u}^T\mathbf{w} = 0$, i.e., $\mathbf{w}$ should be orthogonal to $\mathbf{u}$.

3. (a) Square the matrix $P = aa^T/(a^Ta)$, which projects onto a line, and show that $P^2 = P$.

Hint: Note the number a^Ta in the middle of the matrix aa^Taa^T!

(b) Find the matrix P that projects every vector b in R^3 onto the line in the direction of a and the projection p. Check that error $e = b - p$ is perpendicular to a for the following:
(i) $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $a = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ (ii) $b = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$ and $a = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$

Hint: $p = Pb$ and show that $e^Ta = 0$.

(c) Is the projection matrix P invertible? Why or why not?

Hint: Use the rank of P matrix to support your statement.

Solution:

(a) $P^2 = [aa^T/(a^Ta)] \times [aa^T/(a^Ta)] = a(a^Ta)a^T/[(a^Ta)(a^Ta)] = aa^T/(a^Ta) = P$.
(a^Ta is a scalar)

(b) (i) $P = \frac{1}{14}\begin{bmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{bmatrix}$, $p = Pb = \frac{1}{14}\begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$, $e = \frac{1}{14}\begin{bmatrix} 2 \\ 8 \\ -4 \end{bmatrix}$,

and $e^Ta = \frac{1}{14}\begin{bmatrix} 2 & 8 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 0$.

(ii) $P = \frac{1}{11}\begin{bmatrix} 1 & 3 & -1 \\ 3 & 9 & -3 \\ -1 & -3 & 1 \end{bmatrix}$, $p = Pb = \frac{1}{11}\begin{bmatrix} 4 \\ 12 \\ -4 \end{bmatrix}$, $e = \frac{1}{11}\begin{bmatrix} 7 \\ 21 \\ 70 \end{bmatrix}$,

and $e^Ta = \frac{1}{11}\begin{bmatrix} 7 & 21 & 70 \end{bmatrix}\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} = 0$.

(c) We have proved in tutorial4, that rank(AB) $\leq$ min(rank(A),rank(B)).
So, rank(P) $\leq$ min(rank(a),rank(a^T)) ≤ 1. As matrix P is not a full rank matrix. It is not invertible.

4. A Middle-Aged man was stretched on a rack to lengths $L = 5, 6$, and 7 feet under applied forces of $F = 1, 2$, and 4 tons. Assuming Hooke's law $L = a + bF$, find his normal length a by least squares.

Hint: Least square solution of $Ax = y$ is $\hat{x} = (A^TA)^{-1}A^Ty$. Find A, x, and b in the problem.

> **Solution:** $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$ which is of the form $Ax = y$.
>
> Least square solution is given by:
>
> $(A^TA)^{-1}A^Ty = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 21 & -7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 18 \\ 45 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 63 \\ 9 \end{bmatrix}$
>
> $\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 63 \\ 9 \end{bmatrix}$. So, The normal length a by least squares is 4.5 feet.

5. Let $\mathbf{W} = span(\{(i, 0, 1)\})$ be some subspace in a complex-valued vector space of dimension 3. Find orthonormal bases for $\mathbf{W}$ and $\mathbf{W}^{\perp}$.

Hint: Treat complex-valued vector as any other vector and apply orthogonality principle.

> **Solution:** Since vector space $\mathbf{W}$ is spanned by a single vector, $dim(\mathbf{W}) = 1$. After normalizing, we get orthonormal basis for $\mathbf{W}$ as $\{\frac{1}{\sqrt{2}}(i, 0, 1)\}$. To find a basis for $\mathbf{W}^{\perp}$ is to find a basis for null space of the following system of equations,
>
> $$(a, b, c).(i, 0, 1) = ai + c = 0$$
>
> Thus, $ai = -c$. Now, the basis would be $\{(i, 0, -1), (0, 1, 0)\}$. Notice that, they are already orthogonal ! Finally, normalizing to get orthonormal basis for $\mathbf{W}^{\perp}$ as,
>
> $$\{\frac{1}{\sqrt{2}}(i, 0, -1), (0, 1, 0)\}$$

6. Find the parabola $C + Dt + Et^2$ that comes closest to the values $\mathbf{b} = (0, 0, 1, 0, 0)$ at the times $t = -2, -1, 0, 1, 2$.

Hint: Here, best parabola is symmetric about time t. This directly tells about value of D.

> **Solution:** The five equations $\mathbf{Ax} = \mathbf{b}$ have a rectangular 'Vandermonde' matrix $\mathbf{A}$,
>
> $$A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}, A^TA = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}$$
>
> Note that, column 2 of A is orthogonal to column 1 and 3. The best C, D, E in parabola comes from $\mathbf{A}^T\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^T\mathbf{b}$ and D is uncoupled because its the coefficient of symmetric

parameter t:

$$\begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

leads to $C = 34/70, E = -10/70$ and $D = 0$ (as predicted).

7. If $\mathbf{T}$ is a linear transformation on a vector space $\mathbf{V}$ such that $\|\mathbf{T}(\mathbf{x})\| = \|\mathbf{x}\|$ for $\mathbf{x} \in \mathbf{V}$, prove that it is one to one

   Hint: What happens when $\mathbf{x} \in \mathbf{N}(\mathbf{T})$, where $\mathbf{N}(\mathbf{T})$ is the null space of $\mathbf{T}$?

   Solution: $\mathbf{x} \in \mathbf{N}(\mathbf{T}) \Rightarrow \mathbf{T}(\mathbf{x}) = \mathbf{0}$. Given $\|\mathbf{T}(\mathbf{x})\| = \|\mathbf{x}\| \Rightarrow \|\mathbf{T}(\mathbf{x})\| = \|\mathbf{x}\| = \|\mathbf{0}\| = 0$. Possible iff $\mathbf{x} = \mathbf{0}$. i.e, $\mathbf{N}(\mathbf{T}) = 0$ hence $\mathbf{T}$ is one to one

8. Let $\mathbf{V}$ be an innear product space and define for each pair of vectors $\mathbf{x}, \mathbf{y}$, the scalar $\mathbf{d}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$, called distance between $\mathbf{x}$ and $\mathbf{y}$. Prove for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ that

   (a) $\mathbf{d}(\mathbf{x}, \mathbf{y}) \geq 0$
   (b) $\mathbf{d}(\mathbf{x}, \mathbf{y}) = \mathbf{d}(\mathbf{y}, \mathbf{x})$
   (c) $\mathbf{d}(\mathbf{x}, \mathbf{y}) \leq \mathbf{d}(\mathbf{x}, \mathbf{z}) + \mathbf{d}(\mathbf{z}, \mathbf{y})$
   (d) $\mathbf{d}(\mathbf{x}, \mathbf{x})$=0
   (e) $\mathbf{d}(\mathbf{x}, \mathbf{y}) \neq 0$ if $\mathbf{x} \neq \mathbf{y}$

   Hint: Use the definitions

   Solution:

   (a) norm of a vector is non negative
   (b) $\mathbf{d}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = |-1|\|\mathbf{x} - \mathbf{y}\| = \|(-1)(\mathbf{x} - \mathbf{y})\| = \|\mathbf{y} - \mathbf{x}\| = \mathbf{d}(\mathbf{y}, \mathbf{x})$
   (c) $\mathbf{d}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \|(\mathbf{x} - \mathbf{z}) + (\mathbf{z} - \mathbf{y})\| \leq \|\mathbf{x} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{y}\| = \mathbf{d}(\mathbf{x}, \mathbf{z}) + \mathbf{d}(\mathbf{z}, \mathbf{y})$ by triangular inequality
   (d) $\mathbf{d}(\mathbf{x}, \mathbf{x}) = \|\mathbf{x} - \mathbf{x}\| = \|\mathbf{0}\| = 0$
   (e) Let $\mathbf{x} - \mathbf{y} = \mathbf{z}$, $\|\mathbf{z}\| = 0$ iff $\mathbf{z} = 0 \Rightarrow \mathbf{x} = \mathbf{y}$

9. The fundamental theorem is often stated in the form of *Fredholms alternative*: For any A and $\mathbf{b}$, one and only one of the following systems has a solution:
   (i) $A\mathbf{x} = \mathbf{b}$ (or)
   (ii) $A^T\mathbf{y} = \mathbf{0}$ with $\mathbf{y}^T\mathbf{b} \neq 0$.
   Either $\mathbf{b}$ is in the column space $C(A)$ or there is a $\mathbf{y}$ in $N(A^T)$ such that $\mathbf{y}^T\mathbf{b} \neq 0$. Show that it is contradictory for (i) and (ii) both to have solutions.

   Hint: Proof by contradiction. Start by expressing 0 as $\mathbf{0}^T\mathbf{x}$.

Solution: *Method 1*: Suppose we have both $\mathbf{x}$ satisfies (i) and $\mathbf{y}$ satisfies (ii), then $0 = \mathbf{0}^T\mathbf{x} = (A^T\mathbf{y})^T\mathbf{x} = \mathbf{y}^T(A\mathbf{x}) = \mathbf{y}^T\mathbf{b} \neq 0$.
Therefore either only (i) has a solution or only (ii) has a solution.

Method 2: $C(A)$ and $N(A^T)$ are orthogonal complements in R^m and $C(A^T)$ (row space of A) and $N(A)$ are orthogonal complements in R^n. Now consider the system $A\mathbf{x} = \mathbf{b}$. If $\mathbf{b}$ is in $C(A)$, then a vector $\mathbf{x}$ can be found to satisfy this equation. If $C(A) = R^m$ (i.e. $rank(A) = m$), then there is at least one solution $\mathbf{x}$ for any $\mathbf{b}$. This would also imply that $N(A^T)$ was empty so there could not be a (non-trivial) solution to $A^T\mathbf{y} = \mathbf{0}$. So either there is a solution to $A\mathbf{x} = \mathbf{b}$ or to $A^T\mathbf{y} = \mathbf{0}$. Now given that $N(A^T)$ and $C(A)$ are orthogonal complements, then it is also true that $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is not in $N(A^T)$, i.e. $\mathbf{y}^T\mathbf{b} = 0$.

10. Why is each of these statements false?

    (a) (1,1,1) is perpendicular to (1,1,-2), so the planes $x + y + z = 0$ and $x + y - 2z = 0$ are orthogonal subspaces.

    Hint: Check dimensions of the subspaces.

    (b) The subspace spanned by (1,1,0,0,0) and (0,0,0,1,1) is the orthogonal complement of the subspace spanned by (1,-1,0,0,0) and (2,-2,3,4,-4).

    Hint: Check dimensions of the subspaces.

    (c) Two subspaces that meet only in the zero vector are orthogonal.

    Hint: Take a simple example in 2D space to disprove.

Solution:

(a) Two planes in 3D can not be orthogonal to each other as the sum of their dimensions is greater than 3. (Note that perpendicularity does not automatically imply orthogonality.) These 2 planes meet at a line and if you take any 2 vectors on that line, they are not orthogonal to each other. Orthogonality requires that *every* vector in one subspace be orthogonal to *every* vector in the other subspace.

(b) The first subspace is a 2-D subspace in a 5-D space. Its orthogonal complement must be 3-D, so it cannot be spanned by two vectors.

(c) For example, consider two lines $x = 0$ and $x + y = 0$ on the 2-D plane. These are two lines that intersect at the zero vector, but they are not perpendicular to each other, so they are not orthogonal subspaces.