EE5120 Linear Algebra: Tutorial 4, July-Dec 2017-18

1. State True or False for each of the following with proper justification:
   (a) Let matrix A be a transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then dimension of left nullspace of A, i.e. $N(A^T)$ is $m-r$.
   (b) The pseudoinverse $(A^tA)^{-1}A$ of any linear operator A exists even if the operator is not invertible.
   (c) Let V and W be vector spaces, and let $T: V \to W$ be linear. The T is one-to-one iff $N(T) = \{0\}$.
   (d) Let $v \in \mathbb{R}^n$. The nullity of matrix vv^t is n.
   (e) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a rotation matrix and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a reflection matrix.

2. In $\mathbb{R}^2$, let L be the line $y = 2x$. Find an expression for $\mathbf{T}(x, y)$, where $\mathbf{T}$ is the reflection of $\mathbb{R}^2$ about L.

3. Prove that for two matrices, A, B, the following holds: rank(AB) $\leq$ min(rank(A),rank(B)).

4. Let T be a linear transformation from R^3 into R^2 and U be a linear transformation from R^2 into R^3. Prove that the transformation UT is not invertible. Generalize the theorem. (Can you relate this to question no.7 of the previous tutorial?)

5. What 3 by 3 matrices represent the transformations that,
   (a) project every vector onto the x-y plane?
   (b) reflect every vector through the x-y pane?
   (c) rotate the x-y plane through 90^o, leaving the z-axis alone?
   (d) rotate the x-y plane, then x-z, then y-z through 90^o?
   (e) carry out the same three rotations, but each one through 180^o?

6. Let $\alpha_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ be the (ordered) basis for the vector space $\mathsf{M}_{2\times 2}$, which is the set of all real valued 2×2 matrices. Also, let $\alpha_2 = \{x^2, x, 1\}$ be the basis for the vector space P_2, which is the set of all real polynomials (with real co-efficients) with minimum degree 2. Compute the matrix representations for the following linear transformations:
   (a) $\mathsf{T}_1 : \mathsf{M}_{2\times 2} \to \mathsf{M}_{2\times 2}$ with $\mathsf{T}_1(\mathbf{A}) = \mathbf{A}^T$, for every $\mathbf{A} \in \mathsf{M}_{2\times 2}$.
   (b) $\mathsf{T}_2 : \mathsf{P}_2 \to \mathsf{M}_{2\times 2}$ with $\mathsf{T}_2(f(x)) = \begin{bmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{bmatrix}$. Here, $f'(x)$ and $f''(x)$ are the 1^{st} and 2^{nd} derivatives of $f(x) \in \mathsf{P}_2$.

7. Let V be a vector space and $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be a linear transformation. Suppose $\mathbf{x} \in \mathsf{V}$ is such that $\mathsf{T}^k(\mathbf{x}) = \mathbf{0}$, $\mathsf{T}^m(\mathbf{x}) \neq \mathbf{0}$, $\forall \quad 1 \leq m < k$ and $k > 1$, then prove that the set of vectors $\{\mathbf{x}, \mathsf{T}(\mathbf{x}), \mathsf{T}^2(\mathbf{x}), ..., \mathsf{T}^{k-1}(\mathbf{x})\}$ is linearly independent.

8. BONUS question: **Definition:** Let V be a vector space and $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be a linear transformation on V. A subspace $\mathsf{W} \subset \mathsf{V}$ is said to be T-invariant if for every $\mathbf{w} \in \mathsf{W}$, $\mathsf{T}(\mathbf{w}) \in \mathsf{W}$. Further, if W is T-invariant, define *restriction of T on W* as, $\mathsf{T}_\mathsf{W} : \mathsf{W} \to \mathsf{W}$ such that $\mathsf{T}_\mathsf{W}(\mathbf{w}) = \mathsf{T}(\mathbf{w}), \forall \mathbf{w} \in \mathsf{W}$. Then, prove the following results:
   (a) Subspaces $\{\mathbf{0}\}$, V, N(T) and R(T) are T-invariant. Here, $\mathsf{N(T)} = \{\mathbf{v} \in \mathsf{V} | \mathsf{T}(\mathbf{v}) = \mathbf{0}\}$, and $\mathsf{R(T)} = \{\mathbf{u} \in \mathsf{V} | \exists \mathbf{x_u} \in \mathsf{V} \text{ s.t } \mathsf{T}(\mathbf{x_u}) = \mathbf{u}\}$ (*The choice of* $\mathbf{x_u}$ *depends on* $\mathbf{u}$).
   (b) For a T-invariant subspace W, the transformation T_W is linear, and $\mathsf{N(T_W)} = \mathsf{N(T)} \cap \mathsf{W}$.