EE5120 Linear Algebra: Tutorial 2, July-Dec 2017-18

1. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices, and let $\mathbf{I}_n$ denote an $n \times n$ identity matrix.
   (a) Define a matrix $\mathbf{X}$ as,
   $$\mathbf{X} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ \mathbf{A} & \mathbf{I}_n \end{bmatrix},$$
   where $\mathbf{0}_n$ is an $n \times n$ all zero matrix. Is $\mathbf{X}$ invertible? If so, find the inverse. Else, give proper reason for your answer.
   (b) Trace of a matrix is the sum of diagonal entries of that matrix. Prove that trace of the matrix $\mathbf{AB}$ is equal to that of the matrix $\mathbf{BA}$.

2. Suppose $\mathcal{V}$ is a vector space and $\mathcal{S}_1$, $\mathcal{S}_2$ are subspaces of $\mathcal{V}$. Sum of $\mathcal{S}_1$ and $\mathcal{S}_2$, denoted by $\mathcal{S}_1 + \mathcal{S}_2$ is the set $\{x + y | x \in \mathcal{S}_1, y \in \mathcal{S}_2\}$. Having said that, prove the following:
   (a) $\mathcal{S}_1 + \mathcal{S}_2$ is a subspace of $\mathcal{V}$ that contains both $\mathcal{S}_1$ and $\mathcal{S}_2$.
   (b) Any subspace of $\mathcal{V}$ that contains $\mathcal{S}_1$ and $\mathcal{S}_2$ must also contain $\mathcal{S}_1 + \mathcal{S}_2$.

3. State true or false with proper justifications
   (a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V
   (b) The empty set is a subspace of every vector space
   (c) If V is a vector space and $V \neq \{0\}$, then V contains a subspace W such that $W \neq V$
   (d) The intersection of any two subsets of V is a subspace of V
   (e) An $n \times n$ diagonal matrix can never have more than n non zero entries

4. Determine if the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$ under coordinate wise addition and scalar multiplication. Justify your answers
   (a) $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2,\ a_3 = -a_2\}$
   (b) $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$
   (c) $W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$
   (d) $W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$

5. Given a matrix $\mathbf{A}$,
   $$A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$$
   Find Null space and Column space of $\mathbf{A}$. Present each subspace as a set spanned by a set of linearly independent vectors.

6. Prove this theorem: Let $Ax = b$ be a system of n linear equations in n unknowns. If A is invertible, then the system has exactly one solution, namely $A^{-1}b$. Conversely, if the system has exactly one solution, then A is invertible.
   Hint: Section 2.2 of Gilbert Strang.

7. Suppose A is a 2×1 matrix and that B is a 1×2 matrix. Prove that $C = AB$ is not invertible. Hence prove the general case: if A is an $m \times n$ matrix and B is an $n \times m$ matrix, AB is not invertible if $n < m$.

8. Write down the 3 by 4 finite-difference matrix equation $(h = \frac{1}{5})$ for

$$-\frac{d^2u}{dx^2} + u = x, \quad u(0) = u(1) = 0$$

Solve the equations and compare the solution with the analytical solution. Why are the results not matching and what should be done to improve the accuracy?

9. Use Gaussian elimination without partial pivoting to solve the system of linear euations, rounding to three signnificant digits after each intermediate calculation. Then use partial pivoting to solve the same system, again rounding to three significant digits after each intermediate calculation. Finally, compare both solutions with the given exact solution.

   (a)
   $$x + 1.04y = 2.04,\ 6x + 6.20y = 12.20$$
   (Exact: $x = 1, y = 1$)

   (b)
   $$\begin{bmatrix} 0.143 & 0.357 & 2.01 \\ -1.31 & 0.911 & 1.99 \\ 11.2 & -4.30 & -0.605 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5.173 \\ -5.458 \\ 4.415 \end{bmatrix}$$
   (Exact: $x_1 = 1$, $x_2 = 2$ and $x_3 = -3$).