EE5120 Linear Algebra: Tutorial 2, July-Dec 2017-18

1. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices, and let $\mathbf{I}_n$ denote an $n \times n$ identity matrix.

   (a) Define a matrix $\mathbf{X}$ as,
   $$\mathbf{X} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ \mathbf{A} & \mathbf{I}_n \end{bmatrix},$$
   where $\mathbf{0}_n$ is an $n \times n$ all zero matrix. Is $\mathbf{X}$ invertible? If so, find the inverse. Else, give proper reason for your answer.

   (b) Trace of a matrix is the sum of diagonal entries of that matrix. Prove that trace of the matrix $\mathbf{AB}$ is equal to that of the matrix $\mathbf{BA}$.

Solution:

(a) If $\mathbf{X}$ is invertible, then there must exists a matrix $\mathbf{Y}$ such that
$$\mathbf{XY} = \mathbf{YX} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{I}_n \end{bmatrix}$$
. Let $\mathbf{Y}$ be,
$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 \\ \mathbf{Y}_3 & \mathbf{Y}_4 \end{bmatrix},$$
where $\mathbf{Y}_i$, $\forall i = 1, ..., 4$ be $n \times n$ matrices. To satisfy the first equation, $\mathbf{Y}$ must be such that,
$$\begin{aligned} \mathbf{I}_n\mathbf{Y}_1 + \mathbf{0}_n\mathbf{Y}_3 = \mathbf{I}_n &\Rightarrow \mathbf{Y}_1 = \mathbf{I}_n \\ \mathbf{I}_n\mathbf{Y}_2 + \mathbf{0}_n\mathbf{Y}_4 = \mathbf{0}_n &\Rightarrow \mathbf{Y}_2 = \mathbf{0}_n \\ \mathbf{A}\mathbf{Y}_1 + \mathbf{I}_n\mathbf{Y}_3 = \mathbf{0}_n \Rightarrow \mathbf{A} + \mathbf{Y}_3 = \mathbf{0}_n &\Rightarrow \mathbf{Y}_3 = -\mathbf{A} \\ \mathbf{A}\mathbf{Y}_2 + \mathbf{I}_n\mathbf{Y}_4 = \mathbf{Y}_n &\Rightarrow \mathbf{Y}_4 = \mathbf{I}_n. \end{aligned}$$
Hence, $\mathbf{Y}$ is given by,
$$\mathbf{Y} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ -\mathbf{A} & \mathbf{I}_n \end{bmatrix},$$
It can be easily verified that $\mathbf{YX} = \mathbf{XY} = \mathbf{I}_{2n}$. Hence, inverse of $\mathbf{X}$ exists and the above $\mathbf{Y}$ is its inverse.

(b) Let $(\mathbf{Z})_{i,j}$ denoted the $(i,j)^{th}$ element of a matrix $\mathbf{Z}$. Then, we have the following:e
$$\begin{aligned} \text{Trace}(\mathbf{AB}) &= \sum_{i=1}^{n}(\mathbf{AB})_{i,i} = \sum_{i=1}^{n}\sum_{k=1}^{n}\mathbf{A}_{i,k}\mathbf{B}_{k,i} \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n}\mathbf{B}_{k,i}\mathbf{A}_{i,k} = \sum_{k=1}^{n}\sum_{i=1}^{n}\mathbf{B}_{k,i}\mathbf{A}_{i,k} = \sum_{k=1}^{n}(\mathbf{BA})_{k,k} = \text{Trace}(\mathbf{BA}). \end{aligned}$$

2. Suppose $\mathcal{V}$ is a vector space and $\mathcal{S}_1$, $\mathcal{S}_2$ are subspaces of $\mathcal{V}$. Sum of $\mathcal{S}_1$ and $\mathcal{S}_2$, denoted by $\mathcal{S}_1 + \mathcal{S}_2$ is the set $\{x + y | x \in \mathcal{S}_1, y \in \mathcal{S}_2\}$. Having said that, prove the following:

   (a) $\mathcal{S}_1 + \mathcal{S}_2$ is a subspace of $\mathcal{V}$ that contains both $\mathcal{S}_1$ and $\mathcal{S}_2$.

(b) Any subspace of $\mathcal{V}$ that contains $\mathcal{S}_1$ and $\mathcal{S}_2$ must also contain $\mathcal{S}_1 + \mathcal{S}_2$.

Solution:

(a) Let $\mathcal{S}_1 + \mathcal{S}_2 = \mathcal{S}$. We place the following arguments:

(i) Since $\{\mathcal{S}_i\}_{i=1}^2$ are subspaces, $\mathbf{0} \in \mathcal{S}_1, \mathbf{0} \in \mathcal{S}_2 \Rightarrow \mathbf{0} + \mathbf{0} = \mathbf{0} \in \mathcal{S}$.

(ii) Let $\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1 + \mathbf{v}_2 \in \mathcal{S}$. Then, $\mathbf{u}_1, \mathbf{v}_1 \in \mathcal{S}_1$ and $\mathbf{u}_2, \mathbf{v}_2 \in \mathcal{S}_2$. Since $\{\mathcal{S}_i\}_{i=1}^2$ are subspaces, $\mathbf{u}_i + \mathbf{v}_i \in \mathcal{S}_i$, $i = 1, 2$. Thus, $(\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) \in \mathcal{S} \Rightarrow \mathbf{u}_1 + (\mathbf{v}_1 + \mathbf{u}_2) + \mathbf{v}_2 \in \mathcal{S} \Rightarrow \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{v}_1 + \mathbf{v}_2 \in \mathcal{S}$. This is by commutative and associative properties of vectors in vector space $\mathcal{V}$. Thus, $(\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2) \in \mathcal{S}$.

(iii) Let $\mathbf{u} + \mathbf{v} \in \mathcal{S}$ and a be a scalar from the underlying field. Now, $\mathbf{u} \in \mathcal{S}_1$ and $\mathbf{v} \in \mathcal{S}_2$. As $\mathcal{S}_1$ and $\mathcal{S}_2$ are subspaces, $a\mathbf{u} \in \mathcal{S}_1$ and $a\mathbf{v} \in \mathcal{S}_2$. Then, $a\mathbf{u} + a\mathbf{v} = a(\mathbf{u} + \mathbf{v}) \in \mathcal{S}$, using distributive property defined on $\mathcal{V}$.

Thus, $\mathcal{S}$ is a subspace. Further, let $\mathbf{u} \in \mathcal{S}_1$. Since $\mathbf{0} \in \mathcal{S}_2$, $\mathbf{u} + \mathbf{0} = \mathbf{u} \in \mathcal{S}$ (this is because $\mathbf{0}$ is additive identity of $\mathcal{V}$). Thus, $\mathcal{S}_1 \subset \mathcal{S}$. Similarly, $\mathcal{S}_2 \subset \mathcal{S}$ can be proven.

(b) Again, let $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$. Let $\mathcal{W}$ be a subspace of $\mathcal{V}$ such that,

$$\mathcal{S}_1 \subset \mathcal{W}; \mathcal{S}_2 \subset \mathcal{W}. \tag{1}$$

Let $\mathbf{u} + \mathbf{v} \in \mathcal{S}$. Then, $\mathbf{u} \in \mathcal{S}_1$ and $\mathbf{v} \in \mathcal{S}_2$. Equation (1) implies $\mathbf{u}, \mathbf{v} \in \mathcal{W}$. Since $\mathcal{W}$ is a subspace, $\mathbf{u} + \mathbf{v} \in \mathcal{W}$. Thus, $\mathcal{S} \subset \mathcal{W}$.

3. State true or false with proper justifications

(a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V

(b) The empty set is a subspace of every vector space

(c) If V is a vector space and $V \neq \{0\}$, then V contains a subspace W such that $W \neq V$

(d) The intersection of any two subsets of V is a subspace of V

(e) An $n \times n$ diagonal matrix can never have more than n non zero entries

Solution:

(a) False. It depends on the field. Let $V = \mathbb{R}$ and $W = \mathbb{Q}$ (rational numbers). W is a V.S over $\mathbb{Q}$ but not over $\mathbb{R}$ and hence not a subspace of V

(b) False. Subspace must contain zero vector

(c) True. $W = \{0\}$

(d) False. $V = \mathbb{R}$ let $W_1, W_2 \subset \mathbb{R}$, let $W_1 = \{1\}$, and $W_2 = \{2\}$ $W1 \cap W_2 = \phi$, not a V.S

(e) True.

4. Determine if the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$ under coordinate wise addition and scalar multiplication. Justify your answers

(a) $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2,\ a_3 = -a_2\}$

(b) $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$

(c) $W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$

(d) $W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$

Solution:

(a) Yes. It is a line $t(3, 1, -1)$ through origin (0,0,0) hence a subspace

(b) No. It doesnot contain the zero vector (0,0,0)

(c) Yes.It is a plane with normal vector (2,-7,1)

(d) Yes

5. Given a matrix **A**,

$$A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$$

Find Null space and Column space of **A**. Present each subspace as a set spanned by a set of linearly independent vectors.

Solution: First, decompose the matrix **A** to find its linearly independent set of vectors. Use $\mathbf{Ax} = \mathbf{0}$ to find Nullspace of **A**. For, Column space look for linearly independent column of decomposed matrix **A**.

$$NulA = span(\begin{bmatrix} 3 \\ 1 \end{bmatrix}) \quad \text{and} \quad ColA = span(\begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix})$$

6. Prove this theorem: Let $Ax = b$ be a system of n linear equations in n unknowns. If A is invertible, then the system has exactly one solution, namely $A^{-1}b$. Conversely, if the system has exactly one solution, then A is invertible.
Hint: Section 2.2 of Gilbert Strang.

Solution: Suppose that the system has exactly one solution x_p. Let x_n denote the solution set for the corresponding homogeneous $Ax_n = 0$. By theorem, $x = x_p + x_n$, where x denotes the solution set of system of linear equations $Ax = b$, we get $x = x + x_n$. But this is so only if $x_n = \{0\}$. Thus, (Nullspace) $N(A) = \{0\}$, and hence A is invertible.

7. Suppose A is a 2×1 matrix and that B is a 1×2 matrix. Prove that $C = AB$ is not invertible. Hence prove the general case: if A is an $m \times n$ matrix and B is an $n \times m$ matrix, AB is not invertible if $n < m$.

Solution: A matrix of dimension $n \times n$ will not be invertible if its rank is less than n. Let $A = \begin{bmatrix} a \\ b \end{bmatrix}$ and $B = \begin{bmatrix} c & d \end{bmatrix}$. Then $C = AB = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}$.
Performing the row operation $R_2 = R_2 - \frac{b}{a}R_1$ on AB, it reduces to $AB = \begin{bmatrix} ac & ad \\ 0 & 0 \end{bmatrix}$.
Thus AB has only one independent row, i.e., its rank is less than 2 and therefore not invertible.
For the general case, the matrix formed by the multiplication of an $m \times n$ and an $n \times m$ matrix, with dimension $m \times m$, can similarly be reduced to have n independent rows(or columns) if $n < m$ and hence is not invertible.
Alt Proof: Since B is a 'fat' matrix ($n < m$), there will be at least one non-trivial vector in it's nullspace (the number of pivots will be less than the number of variables). Let such a vector be x_0. Thus, $Bx_0 = 0$. Pre-multiplying by A gives us $Cx_0 = ABx_0 = 0$, i.e. we have found a non-trivial vector x_0 in the null space of C. Thus it can't be invertible.

8. Write down the 4 by 4 finite-difference matrix equation ($h = \frac{1}{5}$) for

$$-\frac{d^2u}{dx^2} + u = x, \quad u(0) = u(1) = 0$$

Solve the equations and compare the solution with the analytical solution. Why are the results not matching and what should be done to improve the accuracy?

Solution: Splitting the line segment from $x = 0$ to $x = 1$ into h-length segments gives us: $x_n = nh$, where $n = 1, 2, 3, 4$
Second Difference is given as,

$$\frac{d^2u}{dx^2} = \frac{u_{n-1} - 2u_n + u_{n+1}}{h^2}$$

where, $u_n = u(x_n)$.
Substitute Second Difference into the equation to get,

$$-\frac{-u_{n-1} + 2u_n - u_{n+1}}{h^2} + u_n = x_n$$

Substitute x_n in terms of h and write the equation for $n = 1, 2, 3, 4$, with boundary conditions and rearrange in matrix form as shown below,

$$\begin{bmatrix} 2.04 & -1 & 0 & 0 \\ -1 & 2.04 & -1 & 0 \\ 0 & -1 & 2.04 & -1 \\ 0 & 0 & -1 & 2.04 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.008 \\ 0.016 \\ 0.024 \\ 0.032 \end{bmatrix}$$

Solution of matrix equation is $(u_1, u_2, u_3, u_4) = (0.0286, 0.0503, 0.0581, 0.0442)$.
Analytical solution is

$$u(x) = x - \frac{\sinh(x)}{\sinh(1)},$$

$$\Rightarrow (u_1, u_2, u_3, u_4) = (0.0287, 0.0505, 0.0583; 0.0443).$$

(Solutions are shown by rounding of to 4 digits.)
The solutions are very near to each other. To still improve the accuracy, we need to decrease h i.e, to increase the number of equations.

9. Use Gaussian elimination without partial pivoting to solve the system of linear equations, rounding to three signnificant digits after each intermediate calculation. Then use partial pivoting to solve the same system, again rounding to three significant digits after each intermediate calculation. Finally, compare both solutions with the given exact solution.

(a)

$$x + 1.04y = 2.04,\ 6x + 6.20y = 12.20$$

(Exact: $x = 1, y = 1$)

(b)

$$\begin{bmatrix} 0.143 & 0.357 & 2.01 \\ -1.31 & 0.911 & 1.99 \\ 11.2 & -4.30 & -0.605 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5.173 \\ -5.458 \\ 4.415 \end{bmatrix}$$

(Exact: $x_1 = 1$, $x_2 = 2$ and $x_3 = -3$).

Solution:

(a) Gaussian elimination without partial pivoting:

$$\begin{bmatrix} 1 & 1.04 \\ 6 & 6.20 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.04 \\ 12.2 \end{bmatrix}$$

Subtract 6 times the first row to second row,

$$\begin{bmatrix} 1 & 1.04 \\ 0 & -0.04 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.04 \\ 0 \end{bmatrix}$$

Thus $y = 0$, using back-substitution we have x = 2.04.
Gaussian elimination using partial pivoting:

$$\begin{bmatrix} 1 & 1.04 \\ 6 & 6.20 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.04 \\ 12.2 \end{bmatrix}$$

'6' is the number with largest magnitude in the first column. So, you need to bring it to first row, i.e exchange row one and two.

$$\begin{bmatrix} 6 & 6.20 \\ 1 & 1.04 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12.2 \\ 2.04 \end{bmatrix}$$

Divide First row my 6,

$$\begin{bmatrix} 1 & 1.03 \\ 1 & 1.04 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.03 \\ 2.04 \end{bmatrix}$$

Add -1 times first row to second row,

$$\begin{bmatrix} 1 & 1.03 \\ 0 & 0.01 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.03 \\ 0.01 \end{bmatrix}$$

Thus $y = 1$, using back-substitution we have x = 1.

(b) After roundoff to three digits

$$\begin{bmatrix} 0.143 & 0.357 & 2.01 \\ -1.31 & 0.911 & 1.99 \\ 11.2 & -4.30 & -0.605 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5.17 \\ -5.46 \\ 4.42 \end{bmatrix}$$

Gaussian elimination without partial pivoting:

$$\begin{bmatrix} 1.00 & 2.50 & 14.1 \\ 0.00 & 1.00 & 4.89 \\ 0.00 & 0.00 & 1.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -36.2 \\ -12.6 \\ -2.00 \end{bmatrix}$$

Thus $x_3 = -2.00$, and using back-elimination we obtain $x_2 = -2.82$ and $x_1 = -0.950$. Gaussian elimination using partial pivoting:

$$\begin{bmatrix} 1.00 & -0.384 & -0.0540 \\ 0.00 & 1.00 & 4.90 \\ 0.00 & 0.00 & 1.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.395 \\ -12.7 \\ -3.00 \end{bmatrix}$$

Thus $x_3 = -3.00$, and using back-elimination we obtain $x_2 = 2.00$ and $x_1 = 1.00$.