EE5120 Linear Algebra: Tutorial 1, July-Dec 2017-18

1. Solve the following sets of linear equations using Gaussian elimination

   (a)
$$\begin{aligned} 2x_1 - 2x_2 - 3x_3 &= -2 \\ 3x_1 - 3x_2 - 2x_3 + 5x_4 &= 7 \\ x_1 - x_2 - 2x_3 - x_4 &= -3 \end{aligned}$$

   (b)
$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 5 \\ x_1 + 4x_2 - 3x_3 - 3x_4 &= 6 \\ 2x_1 + 3x_2 - x_3 + 4x_4 &= 8 \end{aligned}$$

Solution:

(a) System after simplifications ⇒

$$\begin{aligned} x_1 - x_2 - 2x_3 - x_4 &= -3 \\ x_3 + 2x_4 &= 4 \\ 4x_3 + 8x_4 &= 16 \end{aligned}$$

Thus solution is $\{(5 + s - 3t, s, 4 - 2t, t) : s, t \in \mathbb{R}\}$

(b) No solutiom

2. State True or False for each of the following with proper justification:

   (a) An elementary matrix is always a square matrix.

   (b) The $n \times n$ identity matrix is an elementary matrix.

   (c) Product of two elementary matrices (each of appropriate dimensions) is an elementary matrix.

   (d) Sum of two elementary matrices of same dimension is also an elementary matrix.

   (e) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then A can be obtained by performing an elementary row operation on B.

   (f) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then B can also be obtained by performing an elementary column operation on A.

Solution:

(a) True. Since every elementary matrix comes from identity matrix which is a square matrix.

(b) True. It can be considered as an elementary matrix performing the operation of multiplying a row (column) by the scalar value 1.

(c) False. Counter example: Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. A and B are elementary matrices representing multiplication of first row by 2 and exchange of rows, respectively. But $AB = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ which is not an elementary matrix.

(d) False. Counter example: Consider the same matrices A and B stated in (c). Now, $A + B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ which is clearly not an elementary matrix.

(e) True. Suppose E is the elementary matrix such that $B = EA$, then $A = E^{-1}B$, where E^{-1} is the inverse of E which is well defined as inverse of elementary matrices exist.

(f) False. Counter example: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. We can obtain B from A by adding one time the first row of A to its second row. But no column operation on A can change the fact that the second row of A has two zeros.

3. What three elimination matrices E_{21}, E_{31}, E_{32} put A into upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1} and E_{21}^{-1} to factor A into LU where $L = E_{32}^{-1}E_{31}^{-1}E_{21}^{-1}$. Find L and U:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

Solution:

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$, and $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$$\Rightarrow U = E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, and $E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

$$\Rightarrow L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

where $A = LU$.

4. Find L and U for the nonsymmetric matrix

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}$$

Find the four conditions on a, b, c, d, r, s, t to get $A = LU$ with four pivots.

Solution: We are going to do the elimination in the following order E_{43}, E_{32}, and E_{21}.

$$E_{43}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{and } E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow U = E_{21}E_{32}E_{43}A = \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & d-t \end{bmatrix}$$

Conditions to have four pivots are: $a \neq 0$, $b \neq r$, $c \neq s$ and $d \neq t$.
And,

$$\Rightarrow L = E_{43}^{-1}E_{32}^{-1}E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

It should be observed that L matrix has elements also in places where row elimination is not done, as the standard order is not followed in elimination.

5. (a) The equation of the line through the following pair of points $(3, -2, 4)^T$ and $(-5, 7, 1)^T$ in $\mathbb{R}^3$ is

$$(x, y, z)^T = (3, -2, 4)^T + t(........, ..., ...)^T$$

where $t \in \mathbb{R}$.

(b) The equation of the plane through the following set of points $(2, -5, -1)^T$, $(0, 4, 6)^T$ and $(-3, 7, 1)^T$ in $\mathbb{R}^3$ is

$$(x, y, z)^T = (2, -5, -1)^T + s(........, ..., ...)^T + t(........, ..., ...)^T$$

where $s, t \in \mathbb{R}$.

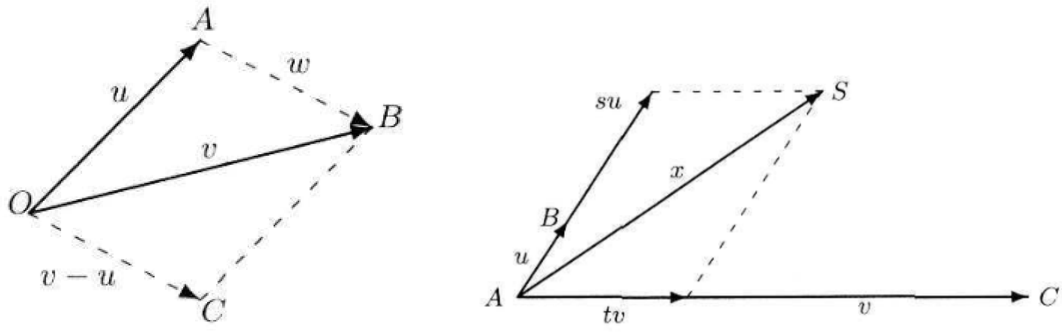
Solution:

(a) Equation of line: (Refer LHS figure below) The endpoint of every vector of the form tw that begins at A lies on the line joining A and B. Thus, an equation of the line through A and B is $x = u + tw = u + t(v - u)$. For the given set of points, equation is

$$(x, y, z)^T = (3, -2, 4)^T + t(-8, 9 - 3)^T$$

(b) Equation of plane: (Refer RHS figure below) For any real number s and t, the vector $su + tv$ lies in the plane containing A, B and C. Thus, the equation of the plane becomes $x = A + su + tv$. For the given set of points, equation is

$$(x, y, z)^T = (2, -5, -1)^T + s(-2, 9, 7)^T + t(-5, 12, 2)^T$$



6. If (a, b) is a multiple of (c, d) with $abcd \neq 0$, show that (a, c) is a multiple of (b, d). Thus show that if a matrix has dependent rows, then it has dependent columns.

Solution: If (a, b) is a multiple of (c, d), then there is some $r \epsilon R$ such that

$$(a, b) = r(c, d) = (rc, rd)$$

Hence,

$$a = rc = r\tfrac{c}{d}d = \tfrac{c}{d}(rd) = \tfrac{c}{d}b.$$

Thus,

$$(a, c) = (\tfrac{c}{d}b, \tfrac{c}{d}d) = \tfrac{c}{d}(b, d)$$

Thus if a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent rows, it has dependent columns.

7. Working with inverse matrices:
   (a) Suppose that $A \in \mathbb{R}^{n \times n}$ is a square invertible matrix and $u, v \in \mathbb{R}^n$ are column vectors. Prove that

   $$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}$$

   provided $1 + v^T A^{-1} u \neq 0$. This formula goes by the name of Sherman-Morrison.
   (b) The practical use of this formula? Let's say that you have been solving matrix equations of the form $Ax = b$ and the system description (embedded in A) changed slightly from A to $A' = A + uv^T$. This formula allows you to use your previous method (e.g. the LU factors of A) in solving for a new $A'x' = b$. Show that a new LU decomposition of A' is unnecessary.

Solution: In part (a) substitute the given inverse and multiply with $A + uv^T A$ to obtain I. In the second part, multiply the given inverse expression with b to see that LU decomposition is not needed afresh.