EE5120 Linear Algebra: Class Test 1, 11th August'17

1. Solve the following sets of linear equations using Gaussian elimination

$$\begin{aligned} x_1 - 4x_2 - x_3 &= -6 \\ 2x_1 - 8x_2 + x_3 &= -9 \\ -x_1 + 4x_2 - 2x_3 &= 3 \end{aligned}$$

Solution: System after simplifications $\Rightarrow$

$$\begin{aligned} x_1 - 4x_2 - x_3 &= -6 \\ 3x_3 &= 3 \end{aligned}$$

Therefore x_3=1; assigning $x_2 = k$ for some $k \in \mathbb{R}$, we have $x_1 = -5 + 4k$.
Thus solution set is $\{(-5, 0, 1) + k(4, 1, 0) : k \in \mathbb{R}\}$

2. A matrix $\mathbf{X}$ is called a projector matrix if $\mathbf{X}^2 = \mathbf{X}$. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors of dimension $N \times 1$ each, and let $\mathbf{I}_N$ denote an $N \times N$ identity matrix. Define $\mathbf{P}_i = \frac{\mathbf{v}_i \mathbf{v}_i^T}{\mathbf{v}_i^T \mathbf{v}_i}$, $i = 1, 2$. Here, $(.)^T$ refers to transpose operation.

   (a) Prove that the matrices $\mathbf{P}_1$, $\mathbf{I}_N - \mathbf{P}_1$ are projector matrices.

   (b) Suppose $\mathbf{v}$ is a vector defined as $\mathbf{v} = (\mathbf{I}_N - \mathbf{P}_1)\mathbf{v}_2$, then compute the expression for $\mathbf{v}_1^T \mathbf{v}$.

Solution:

(a) Note that $\mathbf{v}_1^T \mathbf{v}_1$ is a scalar. Then, we obtain,

$$\mathbf{P}_1^2 = \frac{\mathbf{v}_1 \mathbf{v}_1^T}{\mathbf{v}_1^T \mathbf{v}_1} \frac{\mathbf{v}_1 \mathbf{v}_1^T}{\mathbf{v}_1^T \mathbf{v}_1} = \frac{\mathbf{v}_1 (\mathbf{v}_1^T \mathbf{v}_1) \mathbf{v}_1^T}{(\mathbf{v}_1^T \mathbf{v}_1)^2} = \frac{(\mathbf{v}_1^T \mathbf{v}_1) \mathbf{v}_1 \mathbf{v}_1^T}{(\mathbf{v}_1^T \mathbf{v}_1)^2} = \frac{\mathbf{v}_1 \mathbf{v}_1^T}{\mathbf{v}_1^T \mathbf{v}_1} = \mathbf{P}_1.$$

Hence, $\mathbf{P}_1$ is a projector matrix. Similarly, we can prove $\mathbf{P}_2$ is also projector matrix. Now, consider the following:

$$(\mathbf{I}_N - \mathbf{P}_2)^2 = \mathbf{I}_N^2 + \mathbf{P}_2^2 - 2\mathbf{I}_N \mathbf{P}_2 = \mathbf{I}_N + \mathbf{P}_2 - 2\mathbf{P}_2 = \mathbf{I}_N - \mathbf{P}_2.$$

The second equality uses the fact $\mathbf{P}_2$ is a projector matrix. Hence, $\mathbf{I}_N - \mathbf{P}_2$ is a projector matrix.

(b) Vector $\mathbf{v}$ is given by,

$$\mathbf{v} = (\mathbf{I}_N - \mathbf{P}_1)\mathbf{v}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_1 \mathbf{v}_1^T}{\mathbf{v}_1^T \mathbf{v}_1} \mathbf{v}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_1 (\mathbf{v}_1^T \mathbf{v}_2)}{\mathbf{v}_1^T \mathbf{v}_1}.$$

Now,

$$\mathbf{v}_1^T\mathbf{v} = \mathbf{v}_1^T\left(\mathbf{v}_2 - \frac{\mathbf{v}_1(\mathbf{v}_1^T\mathbf{v}_2)}{\mathbf{v}_1^T\mathbf{v}_1}\right) = \mathbf{v}_1^T\mathbf{v}_2 - \mathbf{v}_1^T\frac{\mathbf{v}_1(\mathbf{v}_1^T\mathbf{v}_2)}{\mathbf{v}_1^T\mathbf{v}_1}$$
$$= \mathbf{v}_1^T\mathbf{v}_2 - \frac{(\mathbf{v}_1^T\mathbf{v}_1)(\mathbf{v}_1^T\mathbf{v}_2)}{\mathbf{v}_1^T\mathbf{v}_1} = \mathbf{v}_1^T\mathbf{v}_2 - \mathbf{v}_1^T\mathbf{v}_2 = 0.$$

Thus, $\mathbf{v}_1^T\mathbf{v} = 0$.