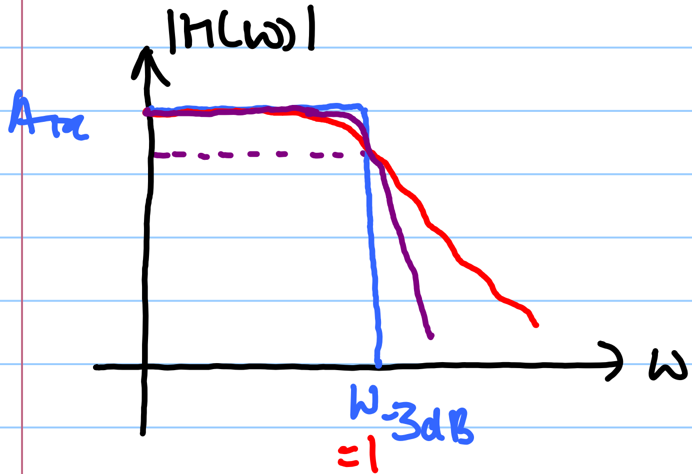
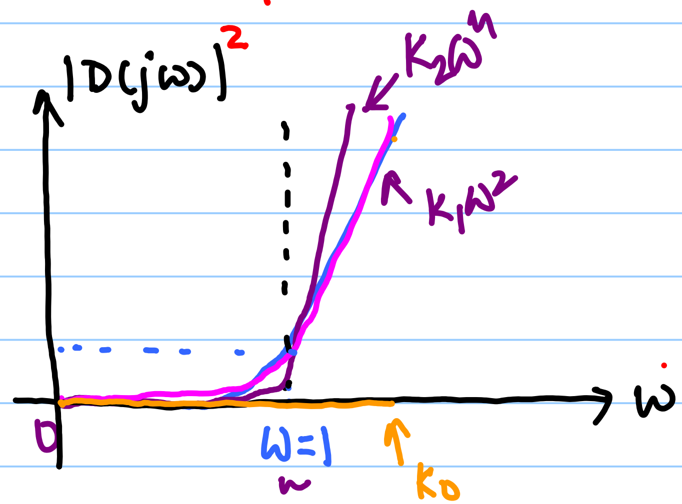
Note Title — 29-04-2021

lecture # 41

Butterworth filter





$$H(s) = \frac{1}{D(s)}$$

$$|H(j\omega)|^2 = \frac{1}{|D(j\omega)|^2} = \frac{1}{K_0 + K_1\omega^2 + \dots}$$

$|D(j\omega)|^2$ = Even fn. of ω.

$$= K_0 + K_1\omega^2 + K_2\omega^4 + \dots + K_n\omega^{2n}$$ ✓

$K_1 = 0,\ K_2 = 0 \ \dots\ K_{n-1} = 0,\ K_n \neq 0$

$$|D(j\omega)|^2 = K_0 + K_n\omega^{2n}$$

$$= K_0\left(1 + \frac{K_n}{K_0}\omega^{2n}\right)$$

DC gain = 1 ⇒ $K_0 = 1$

$\omega = 1$

$$|H(j\omega)|^2 = \frac{1}{1 + \frac{K_n}{1}\omega^{2n}}$$

$$\frac{1}{1 + K_n(\omega_{-3dB})^{2n}} = \frac{1}{2} \Rightarrow \omega_{-3dB} = 1$$

$$|H(j\omega)|^2 = \frac{1}{1+\omega^{2n}}$$ $\Rightarrow$ Maximally flat response within filter bw with $\omega_{-3dB} = 1$.

$$\left.\frac{d^p |D(j\omega)|^2}{d\left((\omega^2)^p\right)}\right|_{\omega=0} = 0 \quad \text{for } p = 1, 2, 3, \ldots, n-1$$ for n^{th} order Butterworth filter.

$$|H(j\omega)|^2 = \frac{1}{1+\omega^{2n}} = H(j\omega)\, H(j\omega)^* = H(j\omega)\, H(-j\omega)$$

$$= H(s)\, H(-s)$$

$$|H(s)|^2 = \frac{1}{1+\left(\frac{s}{j}\right)^{2n}}$$ ✓

✓ $H(s)$: poles in L.H.P

$H(-s)$: poles in R.H.P.

$$1+\left(\frac{s}{j}\right)^{2n} = 1+(-1)^n s^{2n} = 0$$

$$s^{2n} = (-1)^n (-1) = (-1)^n \underline{\cos}(\pi) = (-1)^n e^{j(2p\pi+\pi)}$$

n is even

$$s^{2n} = e^{j(2p\pi+\pi)} \checkmark \qquad p: 0, 1, 2, \; , 2n-1$$

$$s = e^{j\left(\frac{p\pi}{n}+\frac{\pi}{2n}\right)} \qquad \frac{\pi}{2}$$

for $n=2$ $\qquad p = 0, 1, 2, 3$

$$s = e^{j\pi/4}, \; e^{j(\pi/2+\pi/4)}, \; e^{j(\pi+\pi/4)}, \; e^{j(3\pi/2+\pi/4)}$$

R.H.P $\qquad$ L.H.P $\qquad$ L.H.P $\qquad$ R.H.P.

L.H.P poles $\qquad \underbrace{-\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}}_{p_1} \qquad \underbrace{\frac{-1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}}_{p_2}$

$$H(s) = \frac{1}{1+s/\omega_p}$$

$$H(j\omega) = \frac{1}{1+j\omega/\omega_p}$$

$$H^*(j\omega) = \frac{1}{1-j\omega/\omega_p}$$

$$= H^*(s)$$

$$= \left(\frac{1}{1-s/\omega_p}\right)^*$$

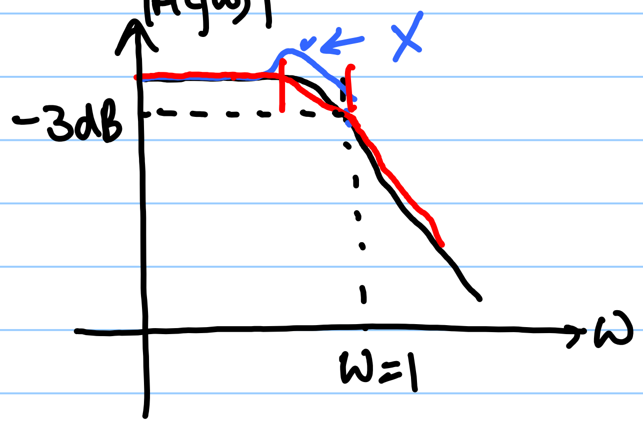
$$H(s) = \frac{1}{(s-p_1)(s-p_2)} = \frac{1}{\left(s+\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

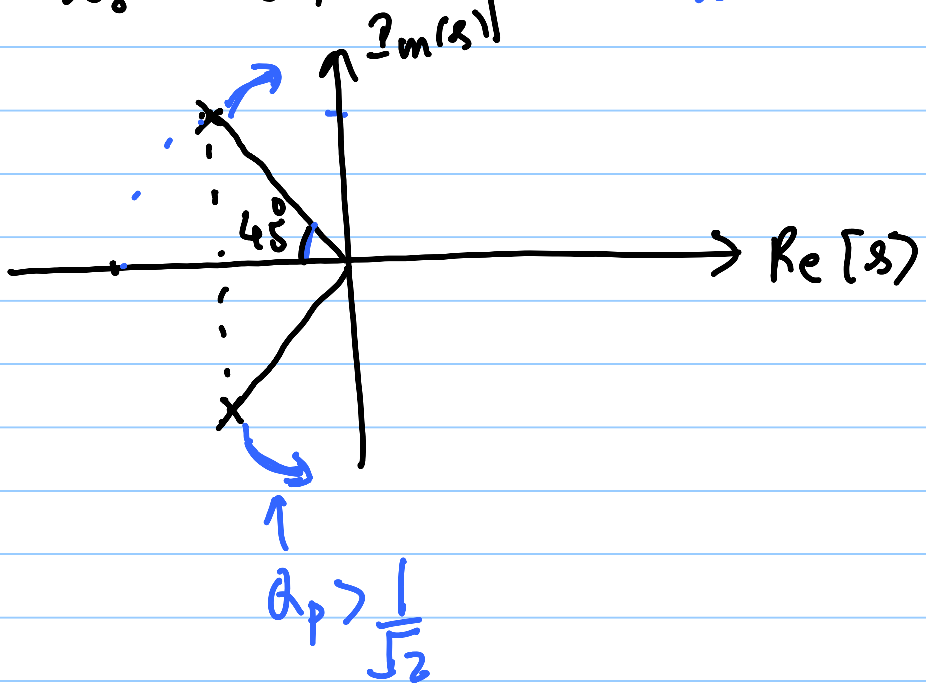
$$= \frac{1}{s^2 + \frac{1}{2} + \sqrt{2}s + \frac{1}{2}}$$

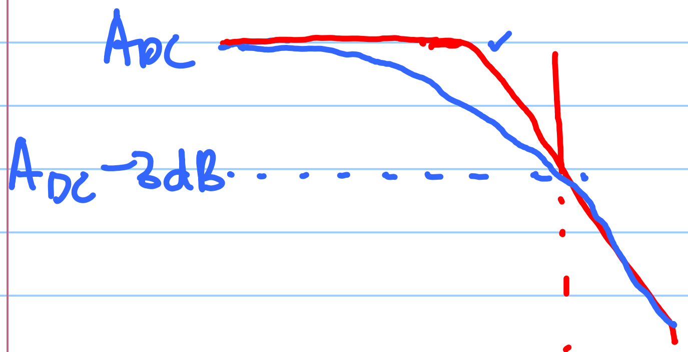
$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1} = \frac{1}{\frac{s^2}{\omega_0^2} + \frac{s}{\omega_0 Q_p} + 1}$$

$\omega_0 = 1$

$Q_p = \frac{1}{\sqrt{2}}$ (Quality factor)







$$H(s) = \frac{1}{1+\sqrt{2}s+s^2} \xrightarrow{s \to s/\omega_0} H(s) = \frac{1}{1+\sqrt{2}\frac{s}{\omega_0}+\frac{s^2}{\omega_0^2}}$$

$$|H(j\omega)|^2 = \frac{1}{1+\underset{\sim}{\omega}^4} \xrightarrow{1 \to \omega_0} |H(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_0}\right)^4}$$

$\omega = 1$ $\qquad\qquad$ $\omega = \omega_0$

Poles: $e^{j\left(\frac{p\pi}{n}+\frac{\pi}{2n}\right)} \longrightarrow \omega_0 e^{j\left(\frac{p\pi}{n}+\frac{\pi}{2n}\right)}$

n as odd

$$s^{2n} = (-1)^n(-1) = 1 = e^{j(2p\pi+2\pi)}$$

$$s = e^{j\left(\frac{p\pi}{n}+\frac{\pi}{n}\right)}$$

$n=3.$

$$s = e^{j\left(\frac{p\pi}{3}+\frac{\pi}{3}\right)} \quad ; \; p = 0,1,2,3,4,5$$

$$s = e^{j\pi/3},\; e^{j2\pi/3},\; e^{j\pi},\; e^{j(\pi+\pi/3)},\; e^{j(5\pi/3)},\; e^{j(2\pi)}$$

R.H.P $\quad$ L.H.P $\quad$ L.H.P $\quad$ L.H.P $\quad$ R.H.P $\quad$ R.H.P

L.H.P poles: $\quad -\frac{1}{2}+j\frac{\sqrt{3}}{2}, \quad -1, \quad -\frac{1}{2}-j\frac{\sqrt{3}}{2}$

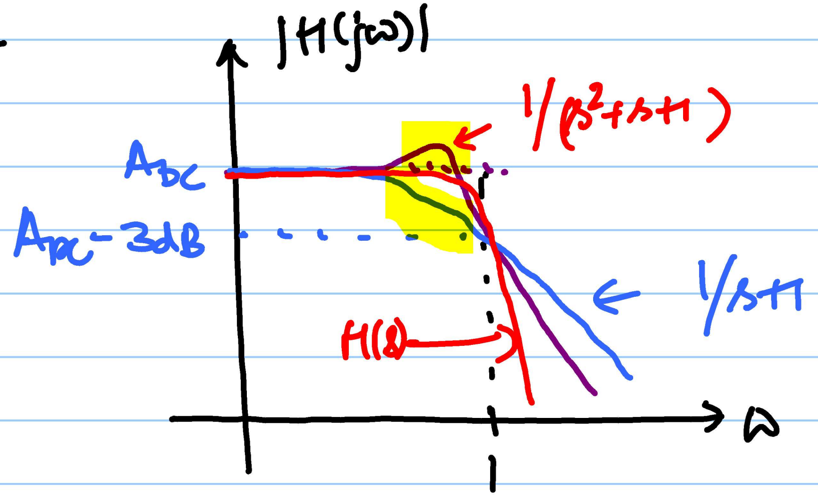
$$H(s) = \frac{1}{(s-p_1)(s-p_2)(s-p_3)}$$

$$= \frac{1}{(s+1)\left\{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right\}}$$

$$= \frac{1}{(s+1)\left(s^2+\frac{1}{4}+s+\frac{3}{4}\right)}$$

$$= \frac{1}{(s+1)\underbrace{(s^2+s+1)}_{Q_p=1}}$$

$$= \frac{1}{s^3+2s^2+2s+1}$$



$$H(s) = \frac{1}{1 + \frac{s}{\omega_0 Q_p} + \frac{s^2}{\omega_0^2}} = \frac{1}{s^2 + \frac{s}{Q_p} + 1}$$

$$|H(j\omega)|^2 \leq 1 \quad \checkmark$$

$$1 \leq (1-\omega^2)^2 + \frac{\omega^2}{Q_p^2}$$

$$\cancel{1} \leq \cancel{1} + \omega^4 - 2\omega^2 + \frac{\omega^2}{Q_p^2}$$

$$0 \leq \omega^2 \left(\omega^2 + \frac{1}{Q_p^2} - 2 \right) \quad \forall \ \omega \in [0, 1]$$

$$\frac{d\,|H(j\omega)|^2}{d\omega} = 0$$

$$\frac{d\left(\frac{1}{(1-\omega^2)^2 + \omega^2/Q_p^2} \right)}{d\omega} = 0$$