Optimum Transmission Strategies for the Gaussian One-to-Many Interference Network

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***Abstract*—We study the Gaussian one-to-many interference network which is obtained as a special case of a general interference network, where only one transmitter generates interference in the network. We allow transmission of messages on *all* the links of the network. This communication model is different from the corresponding one-to-many interference channel. We formulate two transmission strategies for the above network, which involve using Gaussian codebooks and treating interference as noise at a subset of the receivers. We use sum-rate as the criterion of optimality for evaluating the strategies. For the first strategy, we characterize the sum-rate capacity under certain channel conditions, while for the second strategy, we derive a sum-rate outer bound and characterize the gap between the outer bound and the achievable sum-rate of the strategy. Next, we show that the solution approach for the second strategy has applications to the cascade Gaussian Z network, a network consisting of parallel point-to-point links, where each transmitter except the last has a communication link to the adjacent receiver. Lastly, we illustrate the regions corresponding to the derived channel conditions for each strategy.**

keywords: *one-to-many interference network, interference channel, sum capacity.*

I. INTRODUCTION

The one-to-many interference network is a special case of general interference network (IN), where only one transmitter generates interference in the entire network. The system model is shown in Fig. 1. We allow transmission of messages on *all* the links of the network. Without loss of generality, we assume that transmitter 1 generates interference. The communication model assumes that transmitter 1 has an independent message for each receiver, while the other transmitters transmit only to their corresponding receivers. Such a scheme of communication has not been studied before.

The one-to-many interference channel (IC) is a special case of the one-to-many IN, where each transmitter (*Tx* i) is only interested in communicating with its corresponding receiver (*Rx* i), i.e., each transmitter has only one message. The one-to-many IC is studied in [1,2], where the capacity region is characterized to within a constant number of bits. In [3], sum-rate capacity of the one-to-many IC is characterized in the low-interference regime: a regime where using Gaussian inputs and treating interference as noise is optimal.

The one-to-many IN can occur as a communication model both in cellular downlink and uplink as we show below. In cellular downlink, consider the illustration in Fig. 2, where user 1 is within the communication range of base station (BS) 1, whereas users 2 and 3 are at the cell edges of their respective BSs. Users 2 and 3 can receive transmissions from their respective BSs along with BS 1, provided the channel conditions are conducive. In a reverse of the downlink model, in cellular uplink, user 1 is at the cell edge and can transmit to the nearby BSs along with BS 1, while users 2 and 3 communicate with their BSs.

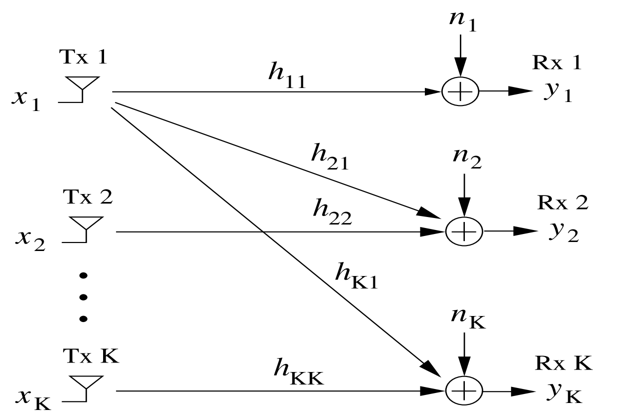

Fig. 1. One-to-many interference network system model with K-transmitters

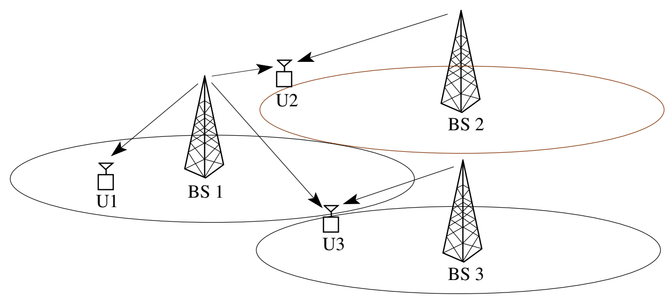

Fig. 2. Illustration of one-to-many interference network in cellular downlink.

For the one-to-many IN, based on the channel conditions, we define the IN to be of either type I or type II. In type I one-to-many IN, receiver 1 can decode all the cross messages from transmitter 1, intended for the other receivers. We show that the sum-rate of a one-to-many IN of type I is the same as that of a one-to-many IC. In type II one-to-many IN, one or more of the receivers can decode the message intended for receiver

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No.	Strategy
$\mathcal{O}1$	Transmitter 1 transmits only to receiver 1 and interference at the other receivers is treated as noise.
$\mathcal{O}2$	Transmitter 1 transmits only to receiver r, $r \neq 1$, and interference at other receivers is treated as noise.

TABLE I
TRANSMISSION STRATEGIES FOR ONE-TO-MANY IN

1. We show that the sum-rate of a one-to-many IN of type II is the same as that the one-to-many IN obtained by eliminating receiver 1. We define the transmission strategies for this network in Table I. All strategies use Gaussian codebooks and interference at a subset of receivers is treated as noise. As for the choice of strategies $\mathcal{O}1$ and $\mathcal{O}2$, we recognize that any transmission from receiver 1 has repercussions at all the receivers. Thus, it is of interest to know when a transmission from receiver 1 to any single receiver is sum-rate optimal.

The analysis of specific transmission strategies for one-to-many IN that are sum-rate optimal is also of interest in the study of half-duplex relay networks [4]. In half-duplex relay networks, the set of transmitters and receivers form an interference network, at any given time instant, including as a special case the one-to-many IN. See [4] for examples of such networks used in optimization of unicast information flow in multistage decode-and-forward relay networks.

The sum-rate at all the receivers is used as the criterion for optimality. We use a 3 transmitter one-to-many IN to analyze the different strategies. For strategy $\mathcal{O}1$, we characterize the sum-rate capacity under certain channel conditions, and for strategy $\mathcal{O}2$, we characterize the gap between the achievable sum-rate and a sum-rate outer bound. Lastly, we show that the solution to strategy $\mathcal{O}2$ has applications to the cascade Gaussian Z network, a network consisting of parallel point-to-point links, where each transmitter except the last has a communication link to the adjacent receiver. As before, we consider messages on all links of this network.

The rest of the paper is organized as follows. The system model is presented in Section II. We discuss the classification of one-to-many IN in Section III. In Section IV, we analyze the two strategies defined earlier. We discuss the applications to the cascade Gaussian Z network in Section V. Some numerical computations regarding the optimality of the strategies are presented in Section VI. Conclusions are presented in Section VII.

II. SYSTEM MODEL

As shown in Fig. 1, the one-to-many IN with K transmitters is characterized by the following input-output equations

$$y_1 = h_{11}\,\tilde{x}_1 + n_1 \tag{1}$$
$$y_i = h_{i1}\,\tilde{x}_1 + h_{ii}\,\tilde{x}_i + n_i, \quad i = 2, 3, \ldots, K, \tag{2}$$

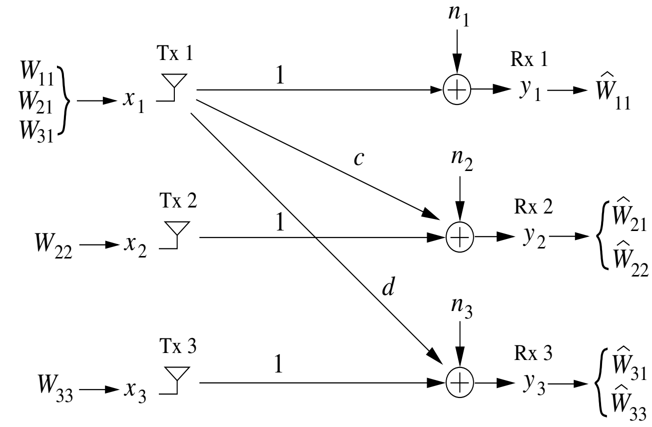

Fig. 3. One-to-many interference network with 3 transmitters in standard form.

where $\tilde{x}_t$ is[1] the transmitted symbol by transmitter t, h_{rt} denotes the complex channel gain from transmitter t to receiver r, and n_r is the additive complex Gaussian noise at the receivers. h_{ii} are known as the direct channels, while h_{i1} are the cross channels, $i = 2, \ldots, K$. The additive noise n_r is a circularly symmetric complex Gaussian (CSCG) random variable with unit variance, i.e., $n_r \sim \mathcal{CN}(0, 1)$, $r = 1, 2, \ldots, K$. Transmitter t is subject to a power constraint $\mathbb{E}[|\tilde{x}_t|^2] \leq \tilde{P}_t$.

We analyze the 3-transmitter one-to-many interference network written in standard form [5]:

$$y_1 = x_1 + n_1 \tag{3}$$
$$y_2 = c\,x_1 + x_2 + n_2 \tag{4}$$
$$y_3 = d\,x_1 + x_3 + n_3, \tag{5}$$

where we have used $c = h_{21} / h_{11}$, $d = h_{31} / h_{11}$, $x_i = h_{ii}\,\tilde{x}_i$, and $P_i = |h_{ii}|^2 \tilde{P}_i$ are the new power constraints. As shown in Fig. 3, the 3-transmitter one-to-many IN has five independent messages, W_{11}, W_{21}, W_{31}, W_{22} and W_{33}, where W_{ij} is the message transmitted from transmitter j to receiver i.

III. CLASSIFICATION OF ONE-TO-MANY IN

We introduce some terminology useful in deriving the results in this section. Let $\mathbf{y}_i^n$ denote the vector of received symbols of length n at receiver i. Let $\mathbf{x}_i^n$ denote the n length vector of transmitted symbols at transmitter i. By Fano's inequality, we have

$$H(W_{ii}\,|\,\mathbf{y}_i^n) \leq n\epsilon_n, \quad i = 1, 2, 3$$
$$H(W_{j1}\,|\,\mathbf{y}_j^n) \leq n\epsilon_n, \quad j = 2, 3,$$

where $\epsilon_n \to 0$ as $n \to \infty$.

A one-to-many IN is said to be stochastically degraded if its conditional marginal distributions are the same as that of a physically degraded IN. Since the error probabilities $\Pr(\hat{W}_{11} \neq W_{11})$ and $\Pr((\hat{W}_{i1}, \hat{W}_{ii}) \neq (W_{i1}, W_{ii}))$ depend only on the conditional marginal distributions $p(y_1|\,x_1)$ and

[1]We use the following notation: lowercase letters for scalars and boldface lowercase letters for vectors. $\bar{[\cdot]}$ denotes complex conjugation and $\mathbb{E}\{\cdot\}$ denotes the expectation operation.

$p(y_i\,|\,x_1, x_i)$, $i = 2, \ldots, K$, the capacity region of the stochastically degraded one-to-many IN is the same as that of the corresponding physically degraded IN.

Definition 1. We define a one-to-many IN to be of *type I* if $|c|^2 \leq 1$ *and* $|d|^2 \leq 1$.

Definition 2. We define a one-to-many IN to be of *type II* if, $|c|^2 \geq 1$, or $|d|^2 \geq 1$ or both.

From the above definitions, it is clear that any general one-to-many IN can be classified into a one-to-many IN of either type I or type II.

A. Implications of type I one-to-many IN

Consider the following set of equations.

$$\begin{aligned} y_2 &= c\,x_1 + x_2 + n_2 \\ &= c\,(x_1 + \tilde{n}_1) + x_2 + \tilde{n}_2, \end{aligned} \tag{6}$$

where $\tilde{n}_1 \sim \mathcal{CN}(0,1)$ and $\tilde{n}_2 \sim \mathcal{CN}(0, 1-|c|^2)$. Since the variance of $\tilde{n}_2$ cannot be negative, we have the necessary condition $|c|^2 \leq 1$. From (6), if $|c|^2 \leq 1$, we have

$$I(W_{21}\,;\,\mathbf{y}_2^n\,|\,\mathbf{x}_2^n) \leq I(W_{21}\,;\,\mathbf{y}_1^n)$$

Therefore, using the independence of $\mathbf{x}_2^n$ and W_{21}, we have

$$\begin{aligned} H(W_{21}\,|\,\mathbf{y}_1^n) &\leq H(W_{21}\,|\,\mathbf{y}_2^n, \mathbf{x}_2^n) \\ &\leq H(W_{21}\,|\,\mathbf{y}_2^n) \leq n\epsilon_n. \end{aligned} \tag{7}$$

Similarly, if $|d|^2 \leq 1$, we have

$$H(W_{31}\,|\,\mathbf{y}_1^n) \leq H(W_{31}\,|\,\mathbf{y}_3^n) \leq n\epsilon_n. \tag{8}$$

From (7) and (8), we have shown that in type I one-to-many IN, receiver 1 can decode W_{21}, W_{31}. The sum-rate can now be bounded as follows

$$\begin{aligned} nS &\leq H(W_{11}, W_{21}, W_{31}) + H(W_{22}) + H(W_{33}) \\ &= I(\mathbf{x}_1^n\,;\,\mathbf{y}_1^n) + H(W_{11}, W_{21}, W_{31}\,|\,\mathbf{y}_1^n) + I(\mathbf{x}_2^n\,;\,\mathbf{y}_2^n) \\ &\quad + H(W_{22}\,|\,\mathbf{y}_2^n) + I(\mathbf{x}_3^n\,;\,\mathbf{y}_3^n) + H(W_{33}\,|\,\mathbf{y}_3^n) \\ &= \sum_{i=1}^{3} I(\mathbf{x}_i^n\,;\,\mathbf{y}_i^n) + H(W_{11}\,|\,\mathbf{y}_1^n) + H(W_{21}\,|\,\mathbf{y}_1^n, W_{11}) \\ &\quad + H(W_{31}|\mathbf{y}_1^n, W_{21}, W_{31}) + H(W_{22}|\,\mathbf{y}_2^n) + H(W_{33}|\,\mathbf{y}_3^n) \\ &\leq \sum_{i=1}^{3} I(\mathbf{x}_i^n\,;\,\mathbf{y}_i^n) + H(W_{11}\,|\,\mathbf{y}_1^n) + H(W_{21}\,|\,\mathbf{y}_1^n) \\ &\quad + H(W_{31}\,|\,\mathbf{y}_1^n) + H(W_{22}\,|\,\mathbf{y}_2^n) + H(W_{33}\,|\,\mathbf{y}_3^n) \qquad (9) \\ &\leq I(\mathbf{x}_1^n\,;\,\mathbf{y}_1^n) + I(\mathbf{x}_2^n\,;\,\mathbf{y}_2^n) + I(\mathbf{x}_3^n\,;\,\mathbf{y}_3^n) + 5\epsilon_n, \qquad (10) \end{aligned}$$

where in (9), we have used the fact that removing conditioning cannot reduce the conditional differential entropy, (10) follows from (7), (8) and Fano's inequality and as $n \to \infty$, $\epsilon_n \to 0$. Note that (10) represents the sum-rate capacity of the one-to-many IC. From (10), we conclude that we can set $W_{21} = W_{31} = \phi$ (without loss of sum-rate).

In summary, the sum-rate capacity for a type I one-to-many IN is same as that of the corresponding one-to-many IC.

B. Implications of type II one-to-many IN

Let $\tilde{y}_2 = c\,x_1 + n_2$. If $|c|^2 \geq 1$, we have

$$\begin{aligned} I(W_{11}\,;\,\mathbf{y}_1^n) &\leq I(W_{11}\,;\,\tilde{\mathbf{y}}_2^n) = I(W_{11}\,;\,c\,\mathbf{x}_1^n + \mathbf{n}_2^n) \\ &= I(W_{11}\,;\,\mathbf{y}_2^n\,|\,\mathbf{x}_2^n). \end{aligned}$$

Thus,

$$H(W_{11}\,|\,\mathbf{y}_2^n, \mathbf{x}_2^n) \leq H(W_{11}\,|\,\mathbf{y}_1^n) \leq n\epsilon_n. \tag{11}$$

From (11), we have shown that if $|c|^2 \geq 1$, receiver 2 can decode W_{11} given $\mathbf{x}_2^n$. Assume $|c|^2 \geq 1$, and further assume that $|c|^2 \geq |d|^2$. Let $\tilde{y}_3 = d\,x_1 + n_3$. Then, we have

$$\begin{aligned} I(W_{31}\,;\,\tilde{\mathbf{y}}_3^n) &\leq I(W_{31}\,;\,\tilde{\mathbf{y}}_2^n) \\ I(W_{31}\,;\,\mathbf{y}_3^n\,|\,\mathbf{x}_3^n) &\leq I(W_{31}\,;\,\mathbf{y}_2^n\,|\,\mathbf{x}_2^n). \end{aligned}$$

Therefore,

$$H(W_{31}\,|\,\mathbf{y}_2^n, \mathbf{x}_2^n) \leq H(W_{31}\,|\,\mathbf{y}_3^n, \mathbf{x}_3^n) \leq n\epsilon_n. \tag{12}$$

The sum-rate can now be bounded as follows

$$\begin{aligned} nS &\leq H(W_{11}, W_{21}, W_{31}, W_{22}) + H(W_{33}) \\ &= I(\mathbf{x}_1^n, \mathbf{x}_2^n\,;\,\mathbf{y}_2^n) + H(W_{11}, W_{21}, W_{31}, W_{22}\,|\,\mathbf{y}_2^n) \\ &\quad + I(\mathbf{x}_3^n\,;\,\mathbf{y}_3^n) + h(\mathbf{x}_3^n\,|\,\mathbf{y}_3^n) \\ &= I(\mathbf{x}_1^n, \mathbf{x}_2^n\,;\,\mathbf{y}_2^n) + h(\mathbf{x}_2^n\,|\,\mathbf{y}_2^n) + H(W_{21}\,|\,\mathbf{y}_2^n, \mathbf{x}_2^n) \\ &\quad + H(W_{11}\,|\,\mathbf{y}_2^n, \mathbf{x}_2^n, W_{21}) + H(W_{31}\,|\,\mathbf{y}_2^n, \mathbf{x}_2^n, W_{11}, W_{21}) \\ &\leq I(\mathbf{x}_1^n, \mathbf{x}_2^n\,;\,\mathbf{y}_2^n) + h(\mathbf{x}_2^n\,|\,\mathbf{y}_2^n) + H(W_{21}\,|\,\mathbf{y}_2^n) \\ &\quad + H(W_{11}\,|\,\mathbf{y}_2^n, \mathbf{x}_2^n) + H(W_{31}\,|\,\mathbf{y}_2^n, \mathbf{x}_2^n) \\ &\quad + I(\mathbf{x}_3^n\,;\,\mathbf{y}_3^n) + h(\mathbf{x}_3^n\,|\,\mathbf{y}_3^n) \qquad (13) \\ &\leq I(\mathbf{x}_1^n, \mathbf{x}_2^n\,;\,\mathbf{y}_2^n) + I(\mathbf{x}_3^n\,;\,\mathbf{y}_3^n) + 5\epsilon_n, \qquad (14) \end{aligned}$$

where in (13), we have used the fact that removing conditioning cannot reduce the conditional entropy, and (14) follows from (11), (12) and Fano's inequality. As $n \to \infty$, $\epsilon_n \to 0$.

Using similar steps as above, it can be shown that when $|d|^2 \geq 1$ and $|d|^2 \geq |c|^2$, the following are true, respectively,

$$\begin{aligned} H(W_{11}\,|\,\mathbf{y}_3^n, \mathbf{x}_3^n) &\leq H(W_{11}\,|\,\mathbf{y}_1^n) \leq n\epsilon_n \\ H(W_{21}\,|\,\mathbf{y}_3^n, \mathbf{x}_3^n) &\leq H(W_{21}\,|\,\mathbf{y}_2^n, \mathbf{x}_2^n) \leq n\epsilon_n, \end{aligned}$$

and receiver 3 can decode W_{11}, W_{21} given $\mathbf{x}_3^n$. The sum-rate in this case is bounded as

$$nS \leq I(\mathbf{x}_2^n\,;\,\mathbf{y}_2^n) + I(\mathbf{x}_1^n, \mathbf{x}_3^n\,;\,\mathbf{y}_3^n) + 5\epsilon_n. \tag{15}$$

From the above arguments, it is clear that, if either $|c|^2 \geq 1$ or $|d|^2 \geq 1$, or both, we can set $W_{11} = \phi$ without any loss in sum-rate. Since W_{11} is the only message intended for receiver 1, this is equivalent to removing receiver 1 from the IN.

In summary, the sum-rate of a one-to-many IN of type II is equivalent to the sum-rate of the channel shown in Fig. 4.

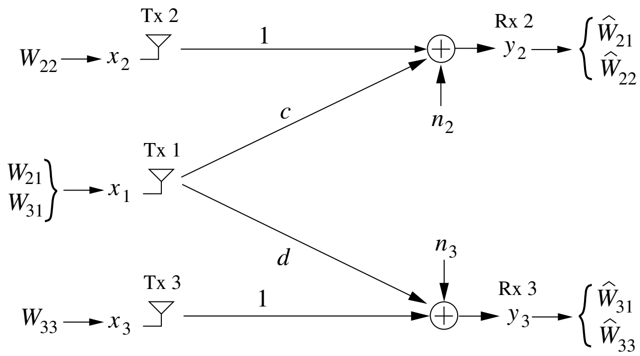

Fig. 4. Equivalent sum-rate representation of type II one-to-many interference network with 3 transmitters

IV. Analysis of Strategies for 3-Transmitter One-to-Many Interference Network

A. Optimality of Strategy $\mathcal{O}1$

Here, we are interested in a region where each transmitter communicates with its corresponding receiver using Gaussian inputs and interference at receivers 2 and 3 is treated as noise. This is referred to in the IC literature as the *low-interference* or the *noisy-interference* regime.

Theorem 1. The sum-rate capacity is achieved by transmitting on the direct channels and treating interference as noise when

$$\frac{|c|^2}{|c|^2\,P_1+1}+\frac{|d|^2}{|d|^2\,P_1+1} \quad\le\quad \frac{1}{P_1+1} \tag{16}$$

Proof: Let $|c|^2 \le 1$, $|d|^2 \le 1$. By definition, this constitutes a one-to-many IN of type I. From Section III-A, we know that the sum-rate of a one-to-many IN of type I is the same as that of a corresponding one-to-many IC. The theorem follows from the characterization of the low-interference regime for the one-to-many IC in [3, Theorem 5]. Lastly, it is not difficult to verify that (16) does not violate the condition for type I one-to-many IN, i.e., $|c|^2 \le 1$ and $|d|^2 \le 1$. □

B. Optimality of Strategy $\mathcal{O}2$

In strategy $\mathcal{O}2$, transmitter 1 communicates solely with either receiver 2 or receiver 3, with the interference at the other receiver treated as noise. Thus, we can equivalently consider the channel shown in Fig. 4. In the following theorem, we derive a sum-rate outer bound for the one-to-many IN and characterize the gap between the outer bound and the sum-rate of strategy $\mathcal{O}2$.

Theorem 2. When transmitter 1 transmits to receiver 2 and interference at receiver 3 is treated as noise, if

$$|c|^2 \ge 1, \quad \text{and} \quad |d|^2 \le \frac{|c|^2|\rho|^2}{(1+P_2)^2}, \tag{17}$$

then the gap between the sum-rate outer bound and the sum-rate of the above strategy is given by

$$\log\left(\frac{1-(1+P_2)^{-1}|\rho|^2}{1-|\rho|^2}\right), \tag{18}$$

where ρ is a constant with $|\rho| \in [0,1]$.

Proof: We assume $|c|^2 \ge 1$ and further that $|c|^2 \ge |d|^2$ in the rest of the proof. Note that this belongs to a type II one-to-many IN and the sum-rate is bounded as (14).

We use genie-aided bounding techniques to derive the optimality of strategy $\mathcal{O}2$. Specifically, we use the concept of *useful genie* and *smart genie* introduced in [3] to obtain an outer bound on the sum-rate of the one-to-many IN. Let a genie provide the following side information to receiver 2:

$$s_2 \quad = \quad d\,x_1+\eta\,z_2, \tag{19}$$

where $z_2 \sim \mathcal{CN}(0,1)$ and η is a positive real number. We allow z_2 to be correlated to n_2 with correlation coefficient ρ.

A genie is said to be useful if it results in a genie-aided channel whose sum-rate capacity is achieved by Gaussian inputs, i.e., the sum-rate capacity of the genie-aided channel equals $I(x_{1G},\,x_{2G}\,;\,y_{2G},s_{2G})+I(x_{3G}\,;\,y_{3G})$, where $x_{iG} \sim \mathcal{CN}(0,P_i)$, y_{iG}, s_{2G} are y_i and s_2 with $x_j = x_{jG}, \forall\, i,j$.

Lemma 1. (Useful Genie) The sum-rate capacity of the genie-aided channel with side information (19) given to receiver 2 is achieved by using Gaussian inputs and by treating interference as noise at other receivers, if the following condition holds:

$$\eta^2 \quad\le\quad 1, \tag{20}$$

and the sum-rate of the genie-aided channel is bounded as

$$S \quad\le\quad I(x_{1G},\,x_{2G}\,;\,y_{2G},s_{2G})+I(x_{3G}\,;\,y_{3G}). \tag{21}$$

Proof: From (14), the sum-rate of the genie-aided channel is bounded as

$$\begin{aligned}
S-5\epsilon_n &\le I(\mathbf{x}_1^n,\,\mathbf{x}_2^n\,;\,\mathbf{y}_2^n,\,\mathbf{s}_2^n)+I(\mathbf{x}_3^n\,;\,\mathbf{y}_3^n)\\
&= I(\mathbf{x}_1^n,\mathbf{x}_2^n;\,\mathbf{s}_2^n)+I(\mathbf{x}_1^n,\mathbf{x}_2^n;\,\mathbf{y}_2^n\,|\,\mathbf{s}_2^n)+I(\mathbf{x}_3^n;\,\mathbf{y}_3^n)\\
&= h(\mathbf{s}_2^n)-h(\mathbf{s}_2^n\,|\,\mathbf{x}_1^n,\,\mathbf{x}_2^n)+h(\mathbf{y}_2^n\,|\,\mathbf{s}_2^n)\\
&\quad -h(\mathbf{y}_2^n\,|\,\mathbf{s}_2^n,\,\mathbf{x}_1^n,\,\mathbf{x}_2^n)+h(\mathbf{y}_3^n)-h(\mathbf{y}_3^n\,|\,\mathbf{x}_3^n)\\
&= h(\mathbf{s}_2^n)-h(\eta\,\mathbf{z}_2^n)+h(\mathbf{y}_2^n\,|\,\mathbf{s}_2^n)\\
&\quad -h(\mathbf{n}_2^n\,|\,\mathbf{z}_2^n)+h(\mathbf{y}_3^n)-h(\mathbf{y}_3^n\,|\,\mathbf{x}_3^n)\\
&= h(\mathbf{s}_2^n)-nh(\eta\,z_2)+nh(y_{2G}\,|\,s_{2G})\\
&\quad -nh(n_2\,|\,z_2)+nh(y_{3G})-h(\mathbf{y}_3^n\,|\,\mathbf{x}_3^n)
\end{aligned} \tag{22}$$

where in (22) we have used Lemma 1 in [3] and the fact that Gaussian inputs maximize the differential entropy for a given covariance constraint.

Thus, it remains to show that $h(\mathbf{s}_2^n)-h(\mathbf{y}_3^n\,|\,\mathbf{x}_3^n)$ is maximized by x_{1G}. Consider the following set of equations,

$$\begin{aligned}
h(\mathbf{s}_2^n)-h(\mathbf{y}_3^n\,|\,\mathbf{x}_3^n) &= h(d\,\mathbf{x}_1^n+\eta\,\mathbf{z}_2^n)-h(d\,\mathbf{x}_1^n+\mathbf{n}_3^n)\\
&\overset{(a)}{\le} nh(d\,x_{1G}+\eta\,z_2)-nh(dx_{1G}+n_3)\\
&= nh(s_{2G})-nh(y_{3G}\,|\,x_{3G}),
\end{aligned}$$

where (a) follows from condition (20) and Lemma 1 in [6], which is a special case of the extremal inequality considered in [7]. Thus, the sum-rate is bounded as (21). □

A smart genie is one which does not increase the sum-rate when Gaussian inputs are used. In this case, since the genie does in fact increase the sum-rate, it is not smart. However, we can choose the parameters ρ and η to get a good sum-rate outer bound as follows. Consider

$$\begin{aligned} & I(x_{1G},\, x_{2G}\,;\, y_{2G},\, s_{2G}) \\ &\quad = \; I(x_{1G},\, x_{2G}\,;\, y_{2G}) + I(x_{1G},\, x_{2G}\,;\, s_{2G}\,|\, y_{2G}). \end{aligned}$$

The second term on the right hand side can be expanded as

$$I(x_{1G}\,;\, s_{2G}\,|\, y_{2G}) + I(x_{2G}\,;\, s_{2G}\,|\, y_{2G},\, x_{1G}).$$

Note that

$$\begin{aligned} I(x_{1G}; s_{2G}\,|\, y_{2G}) &= I(x_{1G}; dx_{1G} + \eta z_2\,|\, cx_{1G} + x_{2G} + n_2) \\ &= I(x_{1G}\,;\, x_{1G} + \eta\, z_2/d\,|\, x_{1G} + (x_{2G} + n_2)/c). \end{aligned}$$

Lemma 8 in [3] says that when x, n, z are Gaussian with x being independent of the two zero-mean random variables n, z, then $I(x\,;\, x+z\,|\, x+n) = 0$, iff $\mathbb{E}(z\,\bar{n}) = \mathbb{E}(|n|^2)$, where $\bar{n}$ denotes the complex conjugate of n. Thus, the above equation reduces to zero if

$$\frac{\eta\,\rho}{d\,\bar{c}} = \frac{1+P_2}{|c|^2} \quad \Rightarrow \quad \eta\,\rho = \frac{d}{c}(1+P_2) \tag{23}$$

Now, consider

$$\begin{aligned} I(x_{2G}; s_{2G}\,|\, y_{2G},\, x_{1G}) &= I(x_{2G}\,;\, \eta\, z_2\,|\, x_{2G} + n_2) \\ &= h(\eta\, z_2\,|\, x_{2G} + n_2) - h(\eta\, z_2\,|\, n_2) \\ &\overset{(a)}{=} h(\eta\, z_2\,|\, x_{2G} + n_2) - h(\eta\, \tilde{z}_2) \\ &= \log\left(\frac{\eta^2(1+P_2) - \eta^2|\rho|^2}{(1+P_2)\eta^2(1-|\rho|^2)}\right) \\ &= \log\left(\frac{1-(1+P_2)^{-1}|\rho|^2}{1-|\rho|^2}\right), \end{aligned} \tag{24}$$

where $\tilde{z}_2 \sim \mathcal{CN}(0, 1-|\rho|^2)$ and (a) follows from [3, Lemma 6]. Note that (24) represents the gap between the sum-rate outer bound and the sum-rate of strategy $\mathcal{O}2$. Eliminating η from (20) and (23), we get $|d|^2 \le |c|^2|\rho|^2/(1+P_2)^2$. □

A similar result can be proved for the case where transmitter 1 transmits to receiver 3 and interference at receiver 2 is treated as noise. We summarize the results for the 3 transmitter one-to-many IN in Table II.

Strat.	Channel conditions	Gap from Outer-bound
$\mathcal{O}1$	$\frac{\lvert c\rvert^2(1+P_1)}{\lvert c\rvert^2 P_1 + 1} + \frac{\lvert d\rvert^2(1+P_1)}{\lvert d\rvert^2 P_1 + 1} \le 1$	0
$\mathcal{O}2$	(i) $\lvert c\rvert^2 \ge 1, \quad \lvert d\rvert^2 \le \frac{\lvert c\rvert^2\lvert\rho\rvert^2}{(1+P_2)^2}$	$\log\left[\frac{1-\frac{\lvert\rho\rvert^2}{1+P_2}}{1-\lvert\rho\rvert^2}\right]$
	(ii) $\lvert d\rvert^2 \ge 1, \quad \lvert c\rvert^2 \le \frac{\lvert d\rvert^2\lvert\rho\rvert^2}{(1+P_3)^2}$	$\log\left[\frac{1-\frac{\lvert\rho\rvert^2}{1+P_3}}{1-\lvert\rho\rvert^2}\right]$

TABLE II
SUMMARY OF RESULTS FOR ONE-TO-MANY INTERFERENCE NETWORK

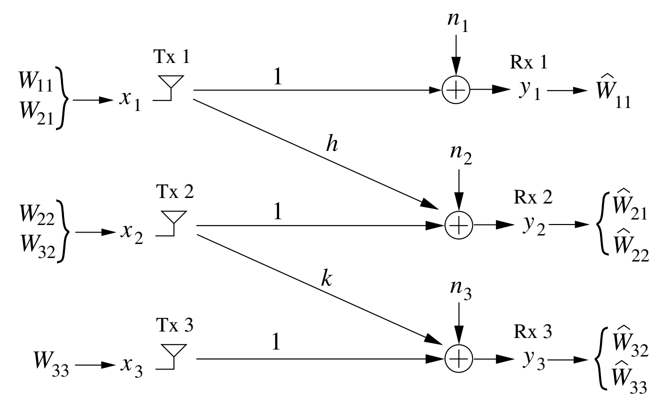

Fig. 5. Cascade Gaussian Z network with 3 transmitters in standard form.

V. Applications to cascade Gaussian Z network

As shown in Fig. 5, we consider the cascade Gaussian Z network with three transmitters, written in standard form. It consists of three parallel point-to-point channels and each transmitter except the last has a communication link to the adjacent receiver. We allow messages on all links of the network.

The cascade Gaussian Z channel is a special case of the above network, where each transmitter is interested in communicating with its corresponding receiver. This channel model is studied in [8]. The cascade Z network with 3 transmitters is characterized by the following input-output equations

$$\begin{aligned} y_1 &= x_1 + n_1 \\ y_2 &= h\,x_1 + x_2 + n_2 \\ y_3 &= k\,x_2 + x_3 + n_3, \end{aligned}$$

It has five independent messages, W_{11}, W_{21}, W_{22}, W_{32} and W_{33}. Transmitter t is subject to a power constraint $\mathbb{E}[|x_t|^2] \le \hat{P}_t$. We define the transmission strategies for this network in Table III.

In strategy $\mathcal{CZ}1$, since transmitters 1 and 2 form a MAC at receiver 2, receiver 1 can be ignored. Likewise, in $\mathcal{CZ}2$, since transmitter 1 transmits to receiver 2, and interference at receiver 1 is treated as noise, receiver 1 can be ignored. Eliminating receiver 1 from Fig. 5 results in the channel shown in Fig. 6, which is similar to the channel shown in Fig. 4. Note that the channel from transmitter 1 to receiver 2 can be normalized to 1 by modifying the power constraint at transmitter 1 from $\hat{P}_1$ to $|h|^2\,\hat{P}_1$. This would make the channel model of Fig. 6 identical to that in Fig. 4. The sum-rate wise optimality of the above strategies can now be bounded by directly applying Theorem 2 to Fig. 6 after bounding the sum-rate.

Thus, we have shown that the solution approach for strategy $\mathcal{O}2$ has applications to the cascade Gaussian Z network.

VI. Numerical results

In this section, we numerically analyze the sum-rate outer bound for the optimality of $\mathcal{O}2$, given in Theorem 2. Let the gap between the sum-rate outer bound and the achievable sum-rate of strategy $\mathcal{O}2$ given in (24) be denoted by Δ. Using (24)

No.	Strategy
$\mathcal{CZ}1$	Transmitters 1 and 2 form a MAC at receiver 2, while interference at receiver 3 is treated as noise.
$\mathcal{CZ}2$	Transmitter 1 transmits to receiver 2, while transmitters 2 and 3 form a MAC at receiver 3 and interference is treated as noise.

TABLE III
TRANSMISSION STRATEGIES FOR CASCADE GAUSSIAN Z NETWORK

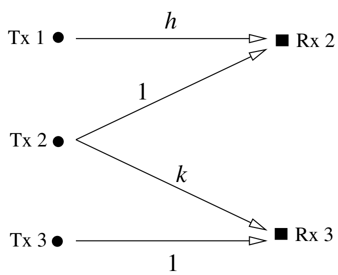

Fig. 6. Equivalent representation of Cascade Gaussian Z network for strategies $\mathcal{CZ}1$ and $\mathcal{CZ}2$.

and solving for ρ in terms of Δ, we get

$$|\rho|^2 = \frac{2^\Delta - 1}{2^\Delta - 1/(1+P_2)} \tag{25}$$

In Fig. 7, we plot $|\rho|^2$ as a function of Δ for different values of P_2. It can be observed that $|\rho|^2$ is a monotonically increasing function of Δ. Thus, to obtain a lower gap from the outer bound, a lower value of $|\rho|^2$ must be chosen. This in turn makes the sub region in (17) smaller. This relationship is explored further is the next figure.

In Fig. 8, we plot the sub region (17) given in Theorem 2 for strategy $\mathcal{O}2$ as a graph in the $|c| - |d|$ plane for various values of Δ, along with the low-interference region (16) for strategy $\mathcal{O}1$. We assume $P_1 = P_2 = P_3 = 0$ dB. As mentioned above, the sub region in (17) shrinks for increasing values of Δ.

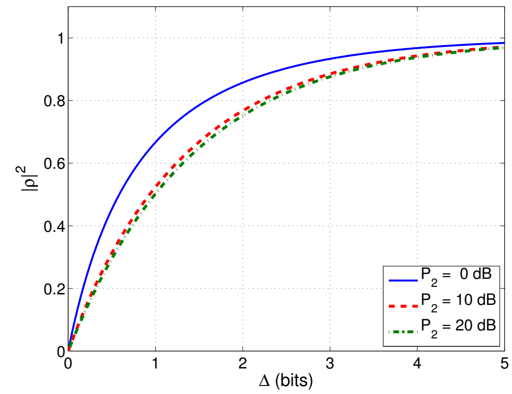

Fig. 7. Variation of $|\rho|^2$ as a function of the gap Δ in bits

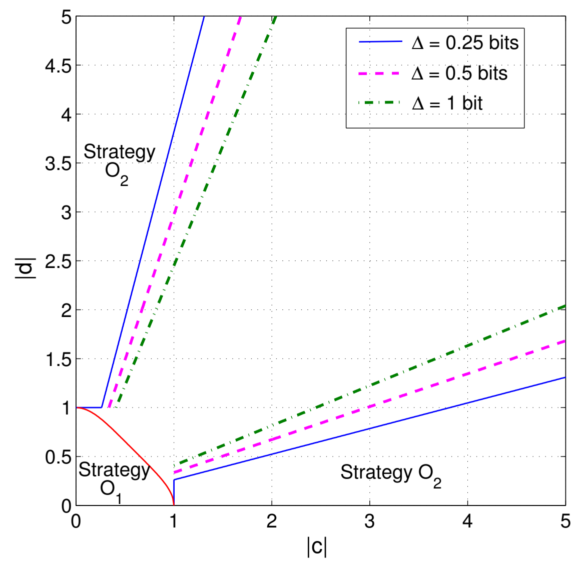

Fig. 8. A plot of the sub regions (16) and (17) for strategies $\mathcal{O}1$, $\mathcal{O}2$, respectively, for $\Delta = 0.25, 0.5$ or 1 bit. $P_1 = P_2 = P_3 = 0$ dB.

VII. CONCLUSIONS

We considered the 3-transmitter Gaussian one-to-many interference network with messages on all the links. We first classified the one-to-many interference network into one-to-many interference network of either type I or type II. Next, we formulated two transmission strategies for this network. Sum-rate capacity was characterized under certain channel conditions for the first strategy whereas for the second strategy, we characterized the gap between the achievable sum-rate and a sum-rate outer bound. Lastly, we showed that the solution approach for the second strategy has applications to the cascade Gaussian Z network.

REFERENCES

[1] A. Jovicic, H. Wang, and P. Viswanath, "On network interference management, " *IEEE Trans. Inform. Theory*, vol. 56, no. 10, pp. 4941–4955, Oct. 2010.

[2] G. Bresler, A. Parekh and D. Tse, "The approximate capacity of the many-to-one and one-to-many Gaussian interference channels," *IEEE Trans. Inform. Theory*, vol. 56, no. 9, pp. 4566–4592, Sep. 2010.

[3] V. S. Annapureddy and V. Veeravalli, "Gaussian interference networks: Sum capacity in the low-interference regime and new outer bounds on the capacity region," *IEEE Trans. Inform. Theory*, vol. 55, no. 7, pp. 3032–3050, Jul. 2009.

[4] B. Muthuramalingam, S. Bhashyam, and A. Thangaraj, "A decode and forward protocol for two-stage Gaussian relay networks," *IEEE Trans. on Comm.*, vol. 60, no. 1, pp. 68–73, Jan. 2012.

[5] A. B. Carleial, "Interference channels", *IEEE Trans. Inform. Theory*, vol. IT-24, pp. 60–70, Jul. 1978.

[6] A. S. Motahari and A. K. Khandani, "Capacity bounds for the Gaussian interference channel," *IEEE Trans. Inform. Theory*, vol. 55, no. 2, pp. 620–643, Feb. 2009.

[7] T. Liu and P. Viswanath, "An extremal inequality motivated by multiterminal information-theoretic problems," *IEEE Trans. Inform. Theory*, vol. 53, no. 5, pp. 1839–1851, May 2007.

[8] Y. Liu and E. Erkip, "On sum capacity of K-user cascade Gaussian Z-interference channel," in *Proc. IEEE ISIT'2011*, St. Petersburg, Russia, Aug. 2011.