On the Sum-Rate of the Gaussian MIMO Z Channel and the Gaussian MIMO X Channel

Ranga Prasad, Srikrishna Bhashyam, *Senior Member, IEEE*, and A. Chockalingam, *Senior Member, IEEE*

***Abstract*—In this paper, we study the Gaussian MIMO Z channel and the Gaussian MIMO X channel. The MIMO X channel (XC) consists of two multiple antenna transmit-receive pairs, where each transmitter communicates with both receivers. The MIMO Z channel (ZC) is obtained from the MIMO X channel by eliminating one of the links and its corresponding message. First, we derive a sum-rate upper bound for the MIMO Z channel and compare it with an existing bound in literature. Next, we consider the MIMO X channel and propose a new sum-rate upper bound by utilizing the sum-rate upper bound for the MIMO ZC. Subsequently, we derive another upper bound for the MIMO XC by assuming receiver cooperation and deriving the worst noise covariance matrix for the resulting two-user MAC. We compare the above two upper bounds for the MIMO XC with the Maddah-Ali-Motahari-Khandani (MMK) scheme. Then, we consider some consequences of the above results for the MIMO interference channel. Finally, we present some numerical results. The numerical results suggest that the proposed sum-rate capacity upper bounds are tighter than existing bounds.**

***Index Terms*—MIMO Z channel, MIMO X channel, MIMO interference channel, sum capacity, upper bound, worst noise covariance.**

I. Introduction

An interesting model to study the effect of interference in communication systems is the single antenna two-user interference channel, consisting of two point-to-point links which interfere with each other. The interference channel (IC) has been widely studied in literature. Although the capacity region of the IC is unknown, several inner and outer bounds for the capacity region and sum-rate capacity have been derived in [1]–[3]. In [3], the capacity region of the IC is characterized to within one bit/s/Hz and in [4]–[6], sum-rate capacity of the IC is characterized in the low-interference regime: a regime where using Gaussian inputs and treating interference as noise is optimal.

Manuscript received May 13, 2014; revised October 14, 2014 and December 10, 2014; accepted December 11, 2014. Date of publication December 22, 2014; date of current version February 12, 2015. Part of this work was presented at the 50th Annual Allerton Conference on Communication, Control, and Computing, Monticello, Illinois, USA, October 2012. The associate editor coordinating the review of this paper and approving it for publication was D. Gunduz.

R. Prasad and A. Chockalingam are with the Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore 560 012, India (e-mail: rprasadn@gmail.com; achockal@ece.iisc.ernet.in).

S. Bhashyam is with the Department of Electrical Engineering, Indian Institute of Technology Madras, Chennai 600 036, India (e-mail: skrishna@ee.iitm.ac.in).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCOMM.2014.2385060

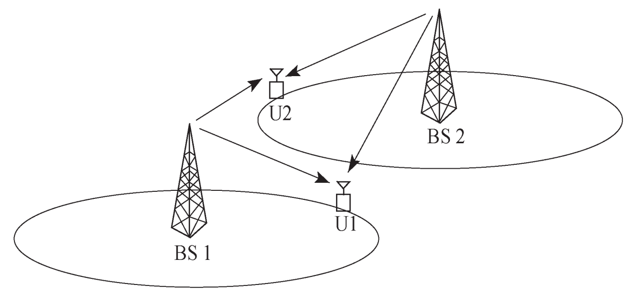

Fig. 1. Illustration of the X channel communication model in the cellular downlink.

A related channel model, the one-sided interference channel, also known as Z-interference channel, is obtained from the IC by removing one of the cross links. Note that in the Z-IC, the cross channel constitutes interference. The Z channel (ZC) is obtained from the Z-IC by considering message transmission on the cross link, and is studied in [7], [8]. By allowing messages on all the links of the IC, we obtain the X channel, i.e., both transmitters have an independent message for each receiver, for a total of four messages in the channel [9], [10]. In this sense, the X channel (XC) is a generalization of the IC.

A. Motivation

The XC can occur as a communication model in cellular networks. Consider the illustration of the cellular downlink in Fig. 1 where user 1 and user 2 are at the cell edges of their respective base stations (BS). Each user can receive transmissions from both base stations. Thus, we have a scenario where the two BSs can communicate independent messages to each user to improve the system throughput. By reversing the direction of transmission, the same model is applicable to the cellular uplink. Another scenario that could be envisaged in the downlink is when large file transfers occur from the BSs to the users, for example, say video files. One solution could be to split the files among the two BSs. Then, both parts of the files could be simultaneously communicated to the users to reduce the download time.

A defining feature of most modern wireless communication systems is the use of multiple antennas at some or all the terminals. Multiple-input multiple-output (MIMO) techniques have attracted attention in wireless communications, since they offer significant increases in data rates without additional bandwidth or increased transmit power. Due to these properties, MIMO technology is an important part of several wireless standards [11]. In this paper, we study three related MIMO channels, namely, the Gaussian MIMO XC, the Gaussian MIMO ZC

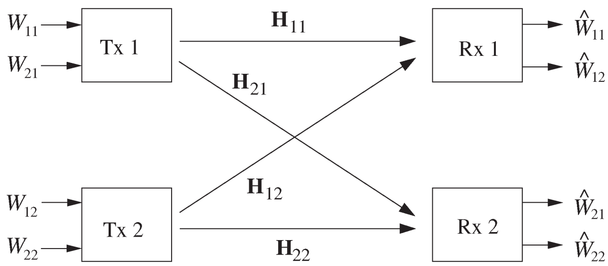

Fig. 2. MIMO X channel.

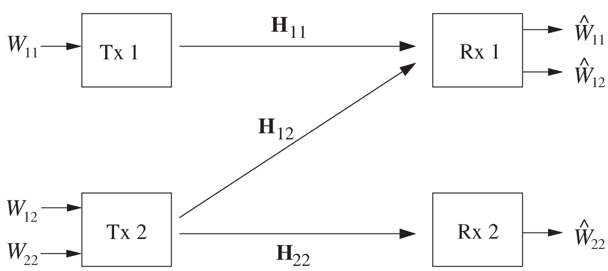

Fig. 3. MIMO Z channel.

and the Gaussian MIMO IC, which are obtained as the MIMO counterparts of the single antenna channels mentioned earlier (see Fig. 2 and Fig. 3).

B. Related Work and Limitations

The sum-rate of the MIMO IC is studied in [12]–[16]. In [12], the capacity region of the MIMO IC is obtained in the so-called *aligned strong* and *very strong interference* regimes. In [12]–[14], the sum-rate of the MIMO IC is characterized in the *noisy-interference* or the *low-interference* regime. The K-user Gaussian MIMO IC is studied in [15], where three upper bounds on the sum-rate are proposed. It is interesting to note that the second bound is in fact the extension of the Etkin-Tse-Wang (ETW) upper bound for the scalar Gaussian IC in [3] to the MIMO case. In a departure from the above results, in [16], the capacity region of the two-user Gaussian MIMO IC is characterized to within a constant gap that is independent of the channel matrices.

The only known upper bound on the sum-rate of the MIMO ZC is obtained in [17, Theorem 1]. This upper bound is based on the idea of reducing the noise at receiver 1 so that it is able to decode all the three messages in the channel resulting in a multiple access channel (MAC). The sum-rate of this MAC, comprising of transmitters 1, 2 and receiver 1, is a sum-rate upper bound for the MIMO ZC.

The MIMO XC has been studied in [17]–[26]. The MIMO XC is especially interesting because of the role it played in the development of the technique of *interference alignment* [18], [19]. Interference alignment refers to the concept of aligning the interfering signals at each receiver, while the desired signals are separable at their respective receivers [17], [20]. The degrees of freedom (DoF) of the MIMO XC was found in [17], and it was shown to be $\frac{4M}{3}$, with $M > 1$ antennas at each node. Interference alignment (IA) was shown to be a key idea for the achievability of the degrees of freedom of the MIMO XC. In [21], the authors combine dirty paper coding, zero forcing (ZF), and successive decoding methods to obtain signaling schemes which achieve the highest multiplexing gain or the degrees of freedom. They eventually transform the XC into four parallel channels. We refer to this scheme as the MMK (Maddah-Ali-Motahari-Khandani) scheme. A gradient projection based IA for the MIMO XC is developed in [22]. Algebraic expressions are derived to obtain a locally optimum IA solution with the objective of maximizing a utility function of the transmit rates. In [23], linear IA transmit filters and ZF receive filters are designed for the XC, based on generalized singular value decomposition. In [24], the authors propose a perfect IA scheme for the K-user MIMO X network, a system consisting of K transmitters and K receivers, where all transmitters send independent messages to all receivers. Space-time precoders with full diversity and low decoding complexity for the XC are investigated in [25]. The noisy-interference regime for the MIMO IC derived in [13] has been extended to the MIMO XC in [26].

We note that results similar to those available for the scalar Gaussian versions of the IC, ZC, and XC are generally much harder to obtain for the multiple antenna case. For example, the problem of characterizing the exact capacity of a MIMO IC can be a challenging problem even for small and special classes. Indeed, the sum-rate capacity of the MIMO IC in the *noisy-interference* and *aligned-strong* regimes in [12], [13], respectively, are characterized in terms of a matrix equation involving the direct and cross link channel matrices. As pointed out in [16], a negligible fraction of channel matrices satisfy such a matrix equation.

This problem is partially addressed in the information theory literature by calculating the DoF of the concerned MIMO channel. The DoF of a MIMO channel denotes the multiplicative scaling of the sum-rate capacity in the high signal-to-noise ratio (SNR) regime. However, more often than not, we are interested in the behavior of sum-rate capacity at finite SNRs, and these DoF capacity approximations fall short of providing an accurate and reliable prediction of the sum-rate capacity at these SNRs. This point is noted in [27], where they advocate the need for stronger approximations to sum-rate capacity. To obtain such stronger capacity approximations, another approach that is usually adopted is to characterize the capacity region or the sum-rate capacity to within a constant gap, irrespective of the channel coefficients and operating SNRs. In the seminal work by Etkin, Tse, and Wang [3], the capacity of a two-user single-input single-output (SISO) Gaussian IC was characterized to within one bit/s/Hz for all values of the channel parameters. In [27], the authors study the SISO Gaussian XC, where they develop a novel deterministic channel model. For this deterministic model, they obtain an approximately optimal communication scheme, which is then translated to the original Gaussian XC. It is shown that this scheme achieves the sum-rate capacity of the Gaussian XC to within a constant gap. However, extending the results of [27] to the MIMO case is non-trivial. In fact, obtaining a good deterministic channel model for a MIMO channel is hard [28]. The only known constant-gap capacity characterization for a MIMO channel is obtained in [16]. In this context, the study of sum-rate bounds for Gaussian MIMO channels such as the ZC, XC, and the IC, that are applicable for all channel matrices and SNRs is of interest. In this paper, we focus on such sum-rate upper bounds valid for all SNRs.

C. Problems Addressed and Contributions Made

In this paper, we make the following contributions:

1) We first derive a sum-rate upper bound for the Gaussian MIMO ZC, obtained by appropriately reducing the noise at receiver 1, resulting in a MAC upper bound at receiver 1.
2) The above upper bound for the MIMO ZC is utilized to propose a new upper bound on the sum-rate of the Gaussian MIMO XC.
3) We derive a second upper bound for MIMO XC by assuming receiver cooperation and deriving the worst noise covariance matrix for the resulting two-user MAC.
4) We compare the upper bounds with the achievable sum-rate of the MMK scheme [21].
5) We specialize the upper bounds derived for the MIMO XC to the MIMO IC and compare them with the sum-rate upper bounds in [15], [16].

We denote the sum-rate upper bound for the MIMO ZC by S_Z^{out}. Following the approach used in [17], the upper bound, S_Z^{out}, for the MIMO ZC is based on the concept of noise reduction at receiver 1. We obtain a class of sum-rate upper bounds by defining a class of noise covariance matrices after noise reduction. This class of noise covariance matrices is characterized by two conditions. We observe that the sum-rate upper bound in [17] can be obtained as a special case of this class of sum-rate upper bounds. Further, we use a result in matrix theory to simultaneously diagonalize the two conditions, leading to an explicit solution for the noise covariance matrix. We show that the sum-rate capacity of the reduced-noise channel is achieved by the MAC formed by transmitters 1, 2 and receiver 1. Numerical results suggest that the proposed bound is tighter than the bound in [17].

Next, S_Z^{out} is utilized to propose a new upper bound on the sum-rate of the Gaussian MIMO XC. We denote this bound $S_X^{\text{out}-1}$. We observe that a MIMO ZC can be obtained from the MIMO XC by eliminating one message and its corresponding channel from the MIMO XC. There are four different Z channels associated with the XC, depending on which message and its corresponding channel are removed. Note that each of the four MIMO ZCs associated with the MIMO XC defines an upper bound on the sum-rate of the remaining three messages. Subsequently, we make use of the sum-rate upper bounds for these four MIMO ZCs to derive $S_X^{\text{out}-1}$.

A second upper bound for the MIMO XC, denoted $S_X^{\text{out}-2}$, is derived as follows. By assuming cooperation among the receivers, we get a Gaussian MIMO MAC with an individual power constraint at each transmitter. Since the MIMO MAC is a MIMO XC with receiver cooperation, the sum-rate capacity of the MIMO MAC is an upper bound on the sum-rate capacity of the MIMO XC. This upper bound can be further tightened by considering noise correlation among the two receivers. This amounts to finding the worst noise covariance matrix for the MAC which gives a much stronger bound. However, finding the least favorable noise covariance matrix is a non-trivial problem as it involves both a maximization over the input covariance matrices and a minimization over the noise covariance matrices. It is shown that the worst noise covariance matrix is a saddle-point of a zero-sum, two-player convex-concave game, which is solved through a primal-dual interior point method that solves the maximization and the minimization parts of the problem simultaneously [29].

Lastly, we consider some ramifications of the above results for the MIMO IC. A sum-rate upper bound for MIMO IC, $S_I^{\text{out}-1}$, is derived by utilizing the sum-rate upper bounds for the MIMO ZC. We also observe that the second upper bound for the XC, i.e., MIMO MAC with worst noise covariance sum-rate upper bound essentially carries over to MIMO IC. We denote this bound by $S_I^{\text{out}-2}$. We compare $S_I^{\text{out}-1}$ and $S_I^{\text{out}-2}$ with the sum-rate upper bounds in [15], [16] and the achievable scheme in [16].

D. Organization

The rest of this paper is organized as follows. The MIMO XC system model is presented in Section II. The first and second upper bounds for the MIMO XC are developed in Sections III and IV, respectively. The MIMO IC is studied in Section V. Numerical results are discussed in Section VI. Conclusions are presented in Section VII.

Notation: Vectors are denoted by boldface lowercase letters, and matrices are denoted by boldface uppercase letters. $[\cdot]^T$ denotes the transpose operation, $[\cdot]^H$ denotes the Hermitian operation, $\text{Tr}(\cdot)$ denotes the trace operation, and $\mathbb{E}[\cdot]$ denotes the expectation operation. Determinant of a matrix $\mathbf{A}$ is denoted by $|\mathbf{A}|$ and $\mathbf{I}$ denotes the identity matrix. For Hermitian matrices $\mathbf{A}$ and $\mathbf{B}$, we use $\mathbf{A} \succeq \mathbf{B}$ to denote that $\mathbf{A}-\mathbf{B}$ is positive semidefinite, and $\mathbf{A} \preceq \mathbf{B}$ to denote that $\mathbf{B}-\mathbf{A}$ is positive semidefinite.

II. System Model

The MIMO XC system model with two transmitters and two receivers is shown in Fig. 2. Transmitter t is equipped with M_t antennas, $t = 1, 2$, and receiver r is equipped with N_r antennas, $r = 1, 2$. In the MIMO XC, each transmitter communicates an independent message to each receiver. Accordingly, the MIMO XC has four independent messages, $W_{11}, W_{12}, W_{21}, W_{22}$, where W_{ij} is the message transmitted from transmitter j to receiver i. Let $\mathbf{H}_{rt} = [h_{ij}]$ denote the $N_r \times M_t$ channel gain matrix from transmitter t to receiver r, where h_{ij} is the channel gain from the jth transmit antenna, $j = 1, 2, \cdots, M_t$, to the ith receive antenna, $i = 1, 2, \cdots, N_r$. The MIMO XC is characterized by the following input-output equations:

$$\mathbf{y}_1 = \mathbf{H}_{11}\mathbf{x}_1 + \mathbf{H}_{12}\,\mathbf{x}_2 + \mathbf{n}_1$$
$$\mathbf{y}_2 = \mathbf{H}_{21}\mathbf{x}_1 + \mathbf{H}_{22}\,\mathbf{x}_2 + \mathbf{n}_2,$$

where $\mathbf{x}_t \in \mathbb{C}^{M_t \times 1}$ denotes the input vector at transmitter t for $t = 1, 2$, $\mathbf{y}_r \in \mathbb{C}^{N_r \times 1}$ denotes the output vector at receiver r for $r = 1, 2$, and $\mathbf{n}_r \in \mathbb{C}^{N_r \times 1}$ is the circularly symmetric complex Gaussian (CSCG) noise vector at receiver r with zero mean and identity covariance matrix, i.e., $\mathbf{n}_r \sim \mathcal{CN}(0, \mathbf{I})$. The input covariance matrices are denoted by $\mathbf{S}_t = \mathbb{E}[\mathbf{x}_t\ \mathbf{x}_t^H]$, $t = 1, 2$. Transmitter t is subject to a power constraint P_t: $\text{Tr}(\mathbf{S}_t) \leq P_t$.

The total power transmitted by both transmitters is denoted by P_T, i.e., $P_T = P_1 + P_2$.

We assume perfect knowledge of all the channel matrices $\mathbf{H}_{rt}$, for $r, t = 1, 2$, at both transmitters and at both receivers. The channel matrices are assumed to be the result of samplings in a rich scattering environment. As such, we ignore the possibility of the channel matrices being rank deficient.

The MIMO IC is characterized by the same input-output equations as the XC, with the distinction that the cross messages are absent, i.e., $W_{12} = W_{21} = \phi$.

III. Upper Bound for MIMO XC Based on MIMO ZC

In this section, we propose a new upper bound on the sum-rate capacity of the Gaussian MIMO XC. This upper bound is formulated by utilizing sum-rate upper bounds for MIMO ZCs. In the following subsection, we first describe the relationship between the MIMO XC and the MIMO ZC. Next, we derive a new sum-rate upper bound for the MIMO ZC. Subsequently, we make use of this upper bound to formulate a new upper bound for the MIMO XC.

A. Obtaining the MIMO ZC From MIMO XC

Consider the MIMO XC shown in Fig. 2. By setting the message $W_{21} = \phi$ and channel $\mathbf{H}_{21} = 0$, we obtain the MIMO ZC in Fig. 3. Thus, both the message as well as the communication link between transmitter 1 and receiver 2 are absent. There are four different ZCs associated with the XC, depending on which message and its corresponding channel are removed. They are denoted by Z(11), Z(12), Z(21) and Z(22), where $Z(ij)$ denotes the ZC obtained from the XC when W_{ij} and $\mathbf{H}_{ij}$ are removed.

B. A New Sum-Rate Upper Bound for the MIMO ZC

In this subsection, we derive a new sum-rate upper bound for the MIMO ZC. The upper bound is based on the concept of reducing the noise at receiver 1 so that it can decode all the three messages in the channel, resulting in a MAC sum-rate upper bound. We state this bound below.

Theorem 1: If $N_1 \geq M_2$ and $N_2 \geq M_2$, then for the MIMO ZC: (i) the sum-rate capacity is bounded by the sum-rate of the MAC formed by transmitter 1, transmitter 2 and receiver 1, with the additive Gaussian noise at receiver 1 modified to $\mathcal{CN}(0, \mathbf{A})$, where

$$\mathbf{A} = \mathbf{I} - \mathbf{H}_{12}\left(\mathbf{H}_{12}^H\mathbf{H}_{12}\right)^{-1}\mathbf{H}_{12}^H + \mathbf{H}_{12}\mathbf{W}\mathbf{H}_{12}^H, \qquad (1)$$

where $\mathbf{W}$ is any $M_2 \times M_2$ positive semidefinite matrix satisfying the following two conditions: $\mathbf{W} \preceq (\mathbf{H}_{12}^H\mathbf{H}_{12})^{-1}$ and $\mathbf{W} \preceq (\mathbf{H}_{22}^H\mathbf{H}_{22})^{-1}$.

(ii) The $M_2 \times M_2$ matrix $\mathbf{W}$ can be chosen as $\mathbf{W} = \mathbf{Q}^{-H}\mathbf{L}\mathbf{Q}^{-1}$, where $\mathbf{Q}$ and $\mathbf{\Lambda}$ are obtained as solutions to the generalized eigenvalue problem

$$\left(\mathbf{H}_{12}^H\mathbf{H}_{12}\right)^{-1}\mathbf{Q} = \left(\mathbf{H}_{22}^H\mathbf{H}_{22}\right)^{-1}\mathbf{Q}\mathbf{\Lambda}, \qquad (2)$$

and $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{M_2})$, $\lambda_i > 0$, $\forall i$. The matrix $\mathbf{L} = \mathrm{diag}(l_1, l_2, \ldots, l_{M_2})$ is a real diagonal matrix, i.e., a diagonal matrix with real diagonal entries, whose elements are chosen as $l_i = \min(\lambda_i, 1)$. The sum-rate of the MIMO ZC is bounded as

$$S_Z \leq S_Z^{\mathrm{out}} = \max_{\substack{\mathrm{Tr}(\mathbf{S}_i) \leq P_i \\ i=1,2}} \log \frac{\left|\mathbf{H}_{11}\mathbf{S}_1\mathbf{H}_{11}^H + \mathbf{H}_{12}\mathbf{S}_2\mathbf{H}_{12}^H + \mathbf{A}\right|}{|\mathbf{A}|}. \qquad (3)$$

Proof: See Appendix B for the proof of statement (i) of Theorem 1. The key idea behind statement (ii) lies in using a simultaneous diagonalization result in matrix theory to equivalently write the two constraints on $\mathbf{W}$ in terms of diagonal matrices, and appropriately choosing $\mathbf{W}$ to satisfy these equivalent constraints. We first make use of the following results in matrix theory to transform the two constraints on $\mathbf{W}$:

Lemma 1 [30, Corollary 4.6.12]: If $\mathbf{B}$ and $\mathbf{C}$ be Hermitian matrices with $\mathbf{C}$ positive definite, then there exists a nonsingular matrix $\mathbf{Q}$ such that $\mathbf{Q}^H\mathbf{B}\mathbf{Q} = \mathbf{\Lambda}$ is a real diagonal matrix and $\mathbf{Q}^H\mathbf{C}\mathbf{Q} = \mathbf{I}$.

Lemma 2 [31, Corollary 8.7.2]: The matrices $\mathbf{Q}$ and $\mathbf{\Lambda}$ in Lemma 1 are obtained as solutions to the generalized eigenvalue problem $\mathbf{B}\mathbf{Q} = \mathbf{C}\mathbf{Q}\mathbf{\Lambda}$.

Applying Lemma 1 to the RHS of the two conditions on $\mathbf{W}$, we observe that there exists a nonsingular matrix $\mathbf{Q}$ such that $\mathbf{Q}^H(\mathbf{H}_{12}^H\mathbf{H}_{12})^{-1}\mathbf{Q} = \mathbf{\Lambda}$ is a real diagonal matrix and $\mathbf{Q}^H(\mathbf{H}_{22}^H\mathbf{H}_{22})^{-1}\mathbf{Q} = \mathbf{I}$. Further, since $\mathbf{Q}^H(\mathbf{H}_{12}^H\mathbf{H}_{12})^{-1}\mathbf{Q}$ is a positive definite matrix, $\mathbf{\Lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{M_2})$ is in fact a diagonal matrix with positive real diagonal entries, i.e., $\lambda_i > 0$, $\forall i$. From Lemma 2, $\mathbf{Q}$, $\mathbf{\Lambda}$ are obtained as solutions to the generalized eigenvalue problem given in (2).

Using the above arguments, it is clear that we can write $(\mathbf{H}_{12}^H\mathbf{H}_{12})^{-1} = \mathbf{Q}^{-H}\mathbf{\Lambda}\mathbf{Q}^{-1}$ and $(\mathbf{H}_{22}^H\mathbf{H}_{22})^{-1} = \mathbf{Q}^{-H}\mathbf{Q}^{-1}$. The two conditions on $\mathbf{W}$ can now be written as $\mathbf{W} \preceq \mathbf{Q}^{-H}\mathbf{\Lambda}\mathbf{Q}^{-1}$ and $\mathbf{W} \preceq \mathbf{Q}^{-H}\mathbf{Q}^{-1}$, or equivalently as follows:

$$\mathbf{Q}^H\mathbf{W}\mathbf{Q} \preceq \mathbf{\Lambda} \text{ and } \mathbf{Q}^H\mathbf{W}\mathbf{Q} \preceq \mathbf{I}. \qquad (4)$$

The above two conditions can easily be satisfied by choosing $\mathbf{Q}^H\mathbf{W}\mathbf{Q} = \mathbf{L}$, where $\mathbf{L} = \mathrm{diag}(l_1, l_2, \ldots, l_{M_2})$ is a real diagonal matrix whose elements are chosen as $l_i = \min(\lambda_i, 1)$. Clearly, it is true that $\mathbf{L} \preceq \mathbf{\Lambda}$ and $\mathbf{L} \preceq \mathbf{I}$. We now obtain an explicit solution for $\mathbf{W}$ as $\mathbf{W} = \mathbf{Q}^{-H}\mathbf{L}\mathbf{Q}^{-1}$. It can be verified that this choice of $\mathbf{W}$ indeed satisfies the two conditions in Theorem 1 by multiplying the LHS and RHS of both terms in (4) by $\mathbf{Q}^{-H}$ and $\mathbf{Q}^{-1}$, respectively, and applying Lemma 4 in Appendix A. ■

Remark 1: The two conditions for the matrix $\mathbf{W}$ in Theorem 1 are *sufficient* to ensure that receiver 1 can decode all the three messages in the MIMO ZC, resulting in a MAC upper bound. Thus, we obtain a class of sum-rate upper bounds for every choice of $\mathbf{W}$ satisfying the above two conditions. The choice of $\mathbf{W}$ that leads to the sum-rate upper bound for MIMO ZC obtained in [17, Theorem 1], derived in the context of DoF analysis is given below. Let the eigenvalue decomposition of $(\mathbf{H}_{12}^H\mathbf{H}_{12})^{-1}$ be $\mathbf{U}\mathbf{\Sigma}_{12}\mathbf{U}^H$, where $\mathbf{U}$ is a unitary matrix of size $M_2 \times M_2$ and $\mathbf{\Sigma}_{12}$ is a $M_2 \times M_2$ diagonal matrix of eigenvalues of $(\mathbf{H}_{12}^H\mathbf{H}_{12})^{-1}$. Similarly, let $\mathbf{V}\mathbf{\Sigma}_{22}\mathbf{V}^H$ be the eigenvalue decomposition of $(\mathbf{H}_{22}^H\mathbf{H}_{22})^{-1}$. By setting $\mathbf{W} = \alpha\mathbf{I}$, the two conditions on $\mathbf{W}$ can be satisfied if $\alpha\mathbf{I} \preceq \mathbf{\Sigma}_{12}$ and $\alpha\mathbf{I} \preceq \mathbf{\Sigma}_{22}$. Clearly, the two conditions are satisfied by choosing

$\alpha = \min(\sigma_{\min}[(\mathbf{H}_{12}^H \mathbf{H}_{12})^{-1}], \sigma_{\min}[(\mathbf{H}_{22}^H \mathbf{H}_{22})^{-1}])$, wherein $\sigma_{\min}(\mathbf{X})$ denotes the minimum singular value of $\mathbf{X}$. Thus, the upper bound for MIMO ZC in [17, Theorem 1] is a special case of Theorem 1.

Remark 2: The DoF result derived for the MIMO XC in [17] based on the MIMO ZC bound in [17, Theorem 1] will remain the same for any $\mathbf{W}$ chosen according to statement (i) in Theorem 1, since the DoF depends only on the fact that the bound arises from a MAC at receiver 1.

C. Upper Bound for MIMO XC Based on MIMO ZCs

Let S_X denote the sum-rate capacity of the MIMO XC. Let R_{ij} denote the rate of message W_{ij}. The upper bound for the MIMO XC is based on the idea that the sum-rate of the MIMO ZC obtained from the MIMO XC by removing one link and its associated message forms an upper bound on the sum-rate of the remaining three messages. Specifically, consider the Z(12) channel obtained from the MIMO XC by eliminating $W_{12} = \phi$ and channel $\mathbf{H}_{12} = 0$. The above statement implies that the sum-rate of the remaining three messages, $R_{11} + R_{21} + R_{22}$ is bounded by the sum-rate capacity of Z(12) channel. Since the sum-rate capacity of the Z(12) channel is not known, we use the sum-rate upper bound in Theorem 1, denoted by $S_{Z(12)}^{\text{out}}$, for bounding the rate of the remaining three messages. Thus, by utilizing the sum-rate upper bounds for the four MIMO ZCs associated with the MIMO XC, a new upper bound on the sum-rate of the MIMO XC, $S_X^{\text{out}-1}$ is obtained. The following theorem gives a set of rate inequalities by considering all the $\binom{4}{3}$ combinations of the rate vector $(R_{11}, R_{12}, R_{21}, R_{22})$, corresponding to each of the four ZCs associated with the XC.

Theorem 2: If $\min(N_1, N_2) \geq \max(M_1, M_2)$, then,

$$R_{12} + R_{21} + R_{22} \leq S_{Z(11)}^{\text{out}} \quad (5)$$

$$R_{11} + R_{21} + R_{22} \leq S_{Z(12)}^{\text{out}} \quad (6)$$

$$R_{11} + R_{12} + R_{22} \leq S_{Z(21)}^{\text{out}} \quad (7)$$

$$R_{11} + R_{12} + R_{21} \leq S_{Z(22)}^{\text{out}}. \quad (8)$$

Proof: Consider the rate inequality (7). Consider any achievable scheme for the MIMO XC. First, we pick W_{21} to be a known sequence shared across all transmitters and receivers. Next, knowledge of W_{11} is given to receiver 2. Thus, receiver 2 can cancel out the contribution of transmitter 1's signal from its received signal. This is equivalent to setting $\mathbf{H}_{21} = 0$, so that we obtain the Z(21) channel. Notice that setting W_{21} to a known sequence and providing receiver 2 with W_{11} does not affect the performance of the achievable scheme with respect to the rate of the messages W_{11}, W_{12} and W_{22}. Thus, using the same achievable scheme, the following sum-rate is also achievable on the MIMO Z(21) channel: $R_{11} + R_{12} + R_{22}$. The inequality (7) follows, since $S_{Z(21)}^{\text{out}}$ represents an upper bound on the sum-rate of the MIMO Z(21) channel. The proof for the other rate inequalities (5), (6) and (8) follow from repeating the above arguments for the Z(11), Z(12), and Z(22) channels, respectively. The condition $\min(N_1, N_2) \geq \max(M_1, M_2)$ is required to satisfy the conditions of the upper bound in Theorem 1. ■

The following theorem is a direct consequence of Theorem 2.

Theorem 3: If $\min(N_1, N_2) \geq \max(M_1, M_2)$, then,

$$S_X \leq S_X^{\text{out}-1} = \frac{1}{3}\left[S_{Z(11)}^{\text{out}} + S_{Z(12)}^{\text{out}} + S_{Z(21)}^{\text{out}} + S_{Z(22)}^{\text{out}}\right]. \quad (9)$$

Proof: Consider the sum of the RHS of the rate inequalities (5)–(8). It is clear that each variable is repeated thrice. Thus, the sum-rate of the MIMO XC is bounded as in (9). ■

IV. Upper Bound for MIMO XC Based on MIMO MAC With Worst Noise Covariance

In this section, we take a different approach to the one considered earlier and derive a new sum-rate upper bound. Consider a MIMO XC where both receivers cooperate to form a corresponding MIMO MAC with the same individual power constraint at the transmitters. Let C_{MAC} be the sum-rate capacity of this MIMO MAC. It is clear that $S_X \leq C_{\text{MAC}}$. The above upper bound is in general loose and can be further tightened by assuming noise correlation at both receivers. Note that the capacity region of the XC depends only on the marginal transition probabilities of the channel (i.e., $\text{p}(\mathbf{y}_i|\mathbf{x}_1, \mathbf{x}_2)$) and not on the joint distribution $\text{p}(\mathbf{y}_1, \mathbf{y}_2|(\mathbf{x}_1, \mathbf{x}_2))$. Hence, correlation between the noise vectors at the receivers of the MIMO XC does not affect the MIMO XC capacity region. However, it does affect the sum-rate capacity of the MIMO MAC, which continues to be an upper bound on the sum-rate capacity of the MIMO XC. Let $\mathbf{z} = [\mathbf{n}_1^T \mathbf{n}_2^T]^T$ be the noise vector in the MAC and let $\mathbf{Z} = \mathbb{E}[\mathbf{z}\,\mathbf{z}^H]$ denote the noise covariance matrix. We let $\mathbb{E}[\mathbf{n}_i \mathbf{n}_i^H] = \mathbf{I}$, $i = 1, 2$, and $\mathbb{E}[\mathbf{n}_1 \mathbf{n}_2^H] \triangleq \tilde{\mathbf{X}}$. Define $\mathbb{S}$ to be the set of all positive semidefinite noise covariance matrices satisfying the MAC upper bound conditions, i.e.,

$$\mathbb{S} = \left\{\mathbf{Z} : \mathbf{Z} \succeq 0,\ \mathbf{Z} = \begin{bmatrix} \mathbf{I}_{N_1} & \tilde{\mathbf{X}} \\ \tilde{\mathbf{X}}^H & \mathbf{I}_{N_2} \end{bmatrix}\right\}. \quad (10)$$

Thus, for any $\mathbf{Z} \in \mathbb{S}$, the MIMO MAC sum-rate capacity C_{MAC} is still an upper bound to S_X. Noise correlation between the multiple antennas within a single receiver affects the capacity of the MIMO XC, and hence is not considered.

We further tighten this upper bound by minimizing C_{MAC} over all the admissible noise covariance matrices $\mathbf{Z}$ to get

$$S_X \leq S_X^{\text{out}-2} = \inf_{\mathbf{Z} \in \mathbb{S}} C_{\text{MAC}}. \quad (11)$$

Let $\mathbf{H}_1^T = [\mathbf{H}_{11}^T \mathbf{H}_{21}^T]$ denote the channel from transmitter 1 to the receiver in the MAC. Similarly, $\mathbf{H}_2^T = [\mathbf{H}_{12}^T \mathbf{H}_{22}^T]$. The MAC upper bound (11) can be written as a min-max problem

$$S_X^{\text{out}-2} = \min_{\mathbf{Z}} \max_{\mathbf{S}_1, \mathbf{S}_2} \log \frac{|\mathbf{H}_1 \mathbf{S}_1 \mathbf{H}_1^H + \mathbf{H}_2 \mathbf{S}_2 \mathbf{H}_2^H + \mathbf{Z}|}{|\mathbf{Z}|}$$
$$\text{s.t.} \quad \mathbf{S}_i \succeq 0,\ \mathbf{Z} \in \mathbb{S},\ \text{Tr}(\mathbf{S}_i) \leq P_i,\ i = 1, 2, \quad (12)$$

where the maximization is over the set of input covariance matrices $\mathbf{S}_i$ at transmitter i, and the minimization is over all possible noise covariance matrices in $\mathbb{S}$. The computation of the minimizing noise covariance matrix $\mathbf{Z}$ above is not necessarily easy, even though the objective function in (12) is convex in $\mathbf{Z}$. Observe that (12) is similar in form to the sum-rate capacity problem of the broadcast channel which can be written as a minimax problem [32], [33]. In [32]–[34], the sum-rate capacity problem of the BC was solved by converting it to a single convex minimization problem. However, it was shown in [29] that when $M_1 > 1$ and $M_2 > 1$, the solution approaches in [32]–[35] cannot be used to convert (12) to a single convex minimization problem. Hence, packages such as CVX [36] cannot be used to solve (12). In Appendix C, we develop a primal-dual interior point method to solve the minimax problem (12). Primal-dual interior point methods are a class of interior point methods which simultaneously solve the primal problem and the dual problem [37]. We closely follow the development of the primal-dual interior point method outlined in [37, Section 11.7].

V. MIMO Interference Channel

Here, we consider some implications of the results for the MIMO XC on the MIMO IC.

A. Upper Bound Based on MIMO Z Channels

We note that a sum-rate upper bound for the MIMO IC can be derived by making use of the sum-rate upper bounds for the MIMO ZC. There are however some crucial differences. In the MIMO IC, the cross links always constitute interference. Therefore, removing one of the cross links enlarges the capacity region. To be more precise, by removing one of the links of the MIMO IC, we obtain the MIMO Z-interference channel. Note that the sum-rate of the MIMO ZC is an upper bound on the sum-rate of the MIMO Z-IC, which in turn is an upper bound on the sum-rate of MIMO IC. Thus, unlike in the MIMO XC, the sum-rate of the MIMO ZC forms an upper bound on the sum-rate of the MIMO IC. The sum-rate upper bound, $S_I^{\text{out}-1}$, is characterized in the following theorem.

Theorem 4: If $\min(N_1, N_2) \geq \max(M_1, M_2)$, then the sum-rate of MIMO IC is bounded as

$$S_I \leq S_I^{\text{out}-1} = \min\left[S_{Z(12)}^{\text{out}}, S_{Z(21)}^{\text{out}}\right]. \tag{13}$$

Proof: Note that there are two MIMO ZCs associated with the MIMO IC depending on which of the cross links are removed, i.e., Z(12) and Z(21). The sum-rate of each of these MIMO ZCs forms an upper bound on the sum-rate of the MIMO IC. A sum-rate upper bound for the MIMO IC, $S_I^{\text{out}-1}$, can now be obtained by considering the minimum of the sum-rate upper bounds for these two MIMO ZCs, i.e., $S_{Z(12)}^{\text{out}}$ and $S_{Z(21)}^{\text{out}}$. The condition $\min(N_1, N_2) \geq \max(M_1, M_2)$ is needed to satisfy the conditions of the upper bound in Theorem 1. ∎

B. Upper Bound Based on MIMO MAC With Worst Noise Covariance

In Section IV, we derived an upper bound for the sum-rate of the MIMO XC by considering cooperation among the two receivers and further tightened this upper bound by assuming noise correlation and deriving the worst noise covariance matrix. This upper bound is given by $S_X \leq S_X^{\text{out}-2}$ and is described by the minimax problem given in (12). Since the sum-rate of the MIMO XC forms an upper bound on the sum-rate of the MIMO IC, it is clear that $S_I \leq S_I^{\text{out}-2} = S_X^{\text{out}-2}$, where we have used $S_I^{\text{out}-2}$ to denote the upper bound for the MIMO IC.

Remark 3: The sum-rate upper bounds in [15], [16] are formulated for given input covariance matrices, $\mathbf{S}_1$, $\mathbf{S}_2$. Thus, to compare the upper bounds $S_I^{\text{out}-1}$ and $S_I^{\text{out}-2}$ with those in [15], [16], we consider input covariance constraints: $\mathbf{S}_i \preceq \mathbf{S}_i^*$, $i = 1, 2$ instead of the trace constraint $\text{Tr}(\mathbf{S}_i) \leq P_i$. Below, we consider the implications of the covariance constraints on the upper bounds, $S_I^{\text{out}-1}$ and $S_I^{\text{out}-2}$. Since $S_I^{\text{out}-1}$ is formulated in terms of the MIMO ZC upper bound in Theorem 1, we evaluate the upper bound (3) in Theorem 1 for the above covariance constraints. Note that $\mathbf{H}_{11}\mathbf{S}_1\mathbf{H}_{11}^H + \mathbf{H}_{12}\mathbf{S}_2\mathbf{H}_{12}^H + \mathbf{A} \preceq \mathbf{H}_{11}\mathbf{S}_1^*\mathbf{H}_{11}^H + \mathbf{H}_{12}\mathbf{S}_2^*\mathbf{H}_{12}^H + \mathbf{A}$. Using the identity $|\mathbf{E}| \geq |\mathbf{F}|$, if $\mathbf{E} \succeq \mathbf{F}$ [30, Corollary 7.7.4], the sum-rate upper bound in (3) can be written as

$$S_Z \leq \log\left|\mathbf{H}_{11}\mathbf{S}_1^*\mathbf{H}_{11}^H + \mathbf{H}_{12}\mathbf{S}_2^*\mathbf{H}_{12}^H + \mathbf{A}\right| / |\mathbf{A}|. \tag{14}$$

The second upper bound is given by the minimax problem in (12), with the trace constraints replaced by the covariance constraints. Using similar arguments as above, we rewrite the second upper bound as

$$S_I^{\text{out}-2} = \min_{\mathbf{Z}\in\mathbb{S}} \quad \log\frac{\left|\mathbf{H}_1\mathbf{S}_1^*\mathbf{H}_1^H + \mathbf{H}_2\mathbf{S}_2^*\mathbf{H}_2^H + \mathbf{Z}\right|}{|\mathbf{Z}|}. \tag{15}$$

Equation (15) follows since $\mathbf{S}_i^*$, $i = 1, 2$, maximize the inner maximization of the minimax problem (12) with covariance constraints.

VI. Numerical Results

In this section, we numerically evaluate the proposed upper bounds for the MIMO ZC, MIMO XC, and MIMO IC, and compare them with other upper bounds and achievable schemes in literature.

A. MIMO ZC

The proposed sum-rate upper bound for MIMO ZC is given in (3), and is formulated in terms of the sum-rate capacity of a two-user MAC with channel matrices $\mathbf{H}_{11}$ and $\mathbf{H}_{12}$, and noise covariance matrix $\mathbf{A}$. The matrices $\mathbf{Q}$ and $\mathbf{\Lambda}$ in (2) can be found using the methods in [30, Corollary 4.6.12], [31, Algorithm 8.7.1], or using the MATLAB command `eig` as follows: $[\mathbf{Q}, \mathbf{\Lambda}] = \texttt{eig}((\mathbf{H}_{12}^H\,\mathbf{H}_{12})^{-1}, (\mathbf{H}_{22}^H\,\mathbf{H}_{22})^{-1})$. The optimizing input covariance matrices, $\mathbf{S}_1$, $\mathbf{S}_2$, that maximize the objective function in (3) can be easily found using the algorithm in [38].

We compare the above upper bound with the sum-rate upper bound for the MIMO ZC derived in [17, Theorem 1]. We note that the upper bound in [17] is derived in the context of DoF analysis, and is generally not expected to be tight at finite SNRs. However, since this is the only sum-rate upper bound available for the MIMO ZC, we compare the upper bound obtained in (3) with that in [17]. The sum-rate upper bound in [17, Theorem 1] can be obtained from Theorem 1 by setting $\mathbf{W} = \alpha\mathbf{I}$, where $\alpha = \min(\sigma_{\min}[(\mathbf{H}_{12}^H\mathbf{H}_{12})^{-1}], \sigma_{\min}[(\mathbf{H}_{22}^H\mathbf{H}_{22})^{-1}])$, wherein $\sigma_{\min}(\mathbf{X})$ denotes the minimum singular value of $\mathbf{X}$.

We consider a MIMO ZC with $M = 3$ antennas at the transmitters and receivers and randomly generate the channel

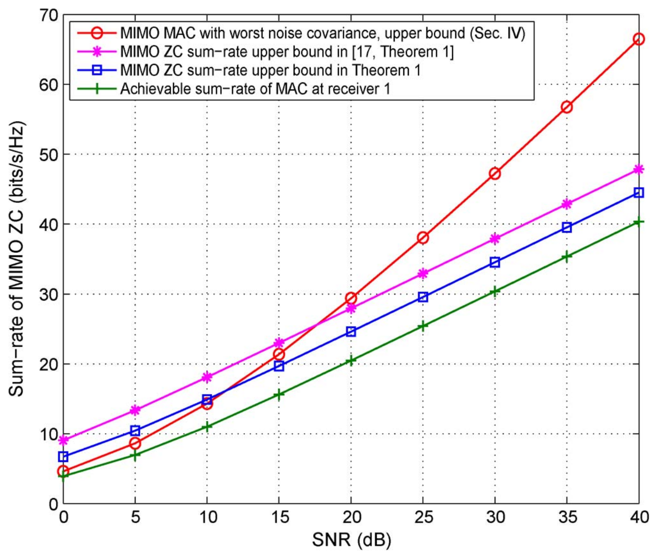

Fig. 4. A plot of the average sum-rates of the proposed upper bound in Theorem 1 and the MIMO ZC upper bound in [17] for 5000 realizations of CSCG channel matrices. $M_1 = M_2 = N_1 = N_2 = 3$.

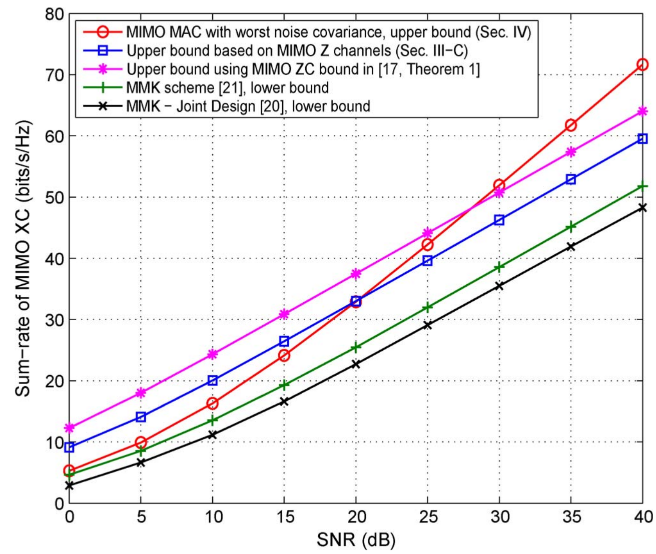

Fig. 5. A plot of the average sum-rates of the proposed upper bounds along with the lower bounds in [20], [21] for a MIMO XC with $M_1 = M_2 = N_1 = N_2 = 3$ for 5000 random realizations of CSCG channel matrices.

matrices, with the channel gain h_{ij} chosen to be a $\mathcal{CN}(0,1)$ random variable, $j = 1, 2, \cdots, M_t$, $i = 1, 2, \cdots, N_r$, $\forall r, t$. We use 5000 realizations of these randomly generated channels and compute the average sum-rates to evaluate the upper bounds. The total power P_T is divided equally between the two transmitters, $P_1 = P_2 = P_T/2$, and the SNR is defined as P_T/σ_n^2, where σ_n^2 is the variance of the CSCG noise at a receive antenna. We plot these results in Fig. 4 and compare the upper bounds in Theorem 1 and [17, Theorem 1]. We also show the achievable sum-rate of the MAC formed by transmitter 1, transmitter 2 and receiver 1 without noise reduction. Also plotted is another upper bound for the MIMO ZC, obtained as a special case of the second upper bound for the MIMO XC, $S_X^{\text{out}-2}$, by setting $\mathbf{H}_{21} = \mathbf{0}$ in Section IV. For the choice of $\mathbf{W}$ in Theorem 1, we observe that the average sum-rate upper bound of Theorem 1 is lower than the average sum-rate upper bound of [17, Theorem 1].

Note that we are unable to prove analytically that the bound in Theorem 1 is tighter than [17]. Since the sum-rate of either bound depends on the noise at receiver 1, it is intuitively clear that the bound with more noise would yield a tighter bound. To gain more insight, we performed the following experiment. We considered a MIMO ZC with $M = 3$ antennas at the transmitters and receivers and generated 1 million realizations of random CSCG channel matrices and compared the sum-rate upper bounds of Theorem 1 and [17, Theorem 1]. For each realization, we consistently observed that the sum-rate upper bound of Theorem 1 was tighter than the sum-rate upper bound of [17, Theorem 1]. We also checked if $\mathbf{W} \succeq \alpha\mathbf{I}$ or $\alpha\mathbf{I} \succeq \mathbf{W}$ was satisfied. We found that $\alpha\mathbf{I} \succeq \mathbf{W}$ was never true, while $\mathbf{W} \succeq \alpha\mathbf{I}$ was true for 0.2% of the channel realizations. For the vast majority of the channel realizations, no such positive definite ordering amongst $\mathbf{W}$ and $\alpha\mathbf{I}$ was observed.

B. MIMO XC

In this subsection, we compare the upper bounds for the MIMO XC, $S_X^{\text{out}-1}$ and $S_X^{\text{out}-2}$, developed in Sections III-C and IV, respectively, with the achievable sum-rate of the MMK scheme [21] in literature. The first upper bound, $S_X^{\text{out}-1}$ can be computed using (9). Each of the terms $S_{Z(ij)}^{\text{out}}$ in (9) can in turn be computed by rewriting the MIMO $Z(ij)$ channel in the standard form shown in Fig. 3 and using (3). The second upper bound, $S_X^{\text{out}-2}$ can be evaluated numerically using the interior point algorithm given in Appendix C. We compare these upper bounds with two closely related schemes, namely, the MMK scheme in [21] and the scheme in [20] called MMK Joint Design, for reference. The sum-rate achieved by the MMK scheme is given by expression (28) in [21] and that of MMK Joint Design can be obtained from the results in [20, Section V].

In Fig. 5, we consider a MIMO XC with $M = 3$ antennas at the transmitters and receivers and plot the average sum-rates of the upper bounds and lower bounds considered above for 5000 realizations of randomly generated channels. Also plotted is the upper bound based on MIMO ZCs in (9), with the terms $S_{Z(ij)}^{\text{out}}$ computed using the upper bound in [17, Theorem 1] instead of that in Theorem 1.

We make the following observations for the plot in Fig. 5. It is seen that the MIMO MAC with worst noise covariance upper bound is quite close to the achievable sum-rate of the MMK scheme at low SNRs, and is moderately close in the medium SNR regime. However, the difference between the bounds increases rapidly, once we approach higher SNRs. Note that with $M = 3$ antennas at each node, the degrees of freedom of this MAC is $2M = 6$, whereas the degrees of freedom of the MIMO XC is $\frac{4M}{3} = 4$. Since the MMK scheme is able to achieve 4 degrees of freedom, there is a difference of 2 degrees of freedom between the MMK scheme and the MAC upper bound. This results in the upper bound being loose in the high SNR regime. However, the upper bound based on ZCs derived in Section III-C, $S_X^{\text{out}-1}$, although loose at low to medium SNRs, is much better at high SNRs. We would like to point out that Fig. 5 is a representative plot for cases where the number of transmit antennas is equal to the number of receive antennas, i.e.,

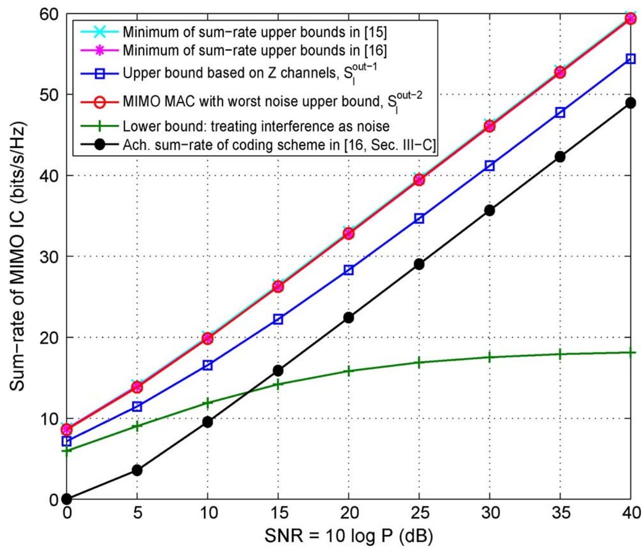

Fig. 6. A plot of the average sum-rates of the proposed upper bounds, along with the upper bounds in [15], [16] for a MIMO IC with $M_1 = M_2 = 2$, $N_1 = N_2 = 4$ for 5000 random realizations of CSCG channel matrices, with the covariance constraints given in (16).

$M_1 = M_2 = N_1 = N_2$. A similar trend is observed for a higher number of antennas at the transmitters and receivers. For example, plots similar to Fig. 5 were obtained for $M = 4$, $N = 4$ and $M = 3$, $N = 4$ as well, where $M_1 = M_2 = M$ and $N_1 = N_2 = N$.

C. MIMO IC

In Fig. 6, we consider a MIMO IC with $M_1 = M_2 = 2$, $N_1 = N_2 = 4$ and evaluate the sum-rate upper bounds for MIMO IC, $S_I^{\text{out}-1}$ and $S_I^{\text{out}-2}$, in (14) and (15), respectively, for 5000 random realizations of CSCG channels with the following covariance constraints:

$$\mathbf{S}_1^* = (P/2)\mathbf{I}, \qquad \mathbf{S}_2^* = \frac{P}{2}\begin{bmatrix} 1 & 0.2+0.2i \\ 0.2-0.2i & 4 \end{bmatrix}, \qquad (16)$$

where $P/2$ denotes power scaling, and compute the average sum-rates, for different values of P.

We also compute the minimum of the sum-rate upper bounds in [15] and [16], respectively. We compare these upper bounds with the achievable sum-rate obtained by treating interference as noise at both receivers. We also plot the achievable sum-rate of the explicit coding scheme in [16, Section III-C]. We see that $S_I^{\text{out}-1}$ is tighter than the other upper bounds and is closer to the achievable sum-rate of the explicit coding scheme in [16, Section III-C].

VII. Conclusion

We investigated the sum-rate capacities of the Gaussian MIMO Z channel, the Gaussian MIMO X channel, and the Gaussian MIMO interference channel. First, we derived a sum-rate upper bound for the MIMO ZC. We then considered the MIMO XC and proposed a new sum-rate upper bound by utilizing the sum-rate upper bound for the MIMO ZC. Subsequently, we derived another upper bound for the MIMO XC by assuming receiver cooperation and deriving the worst noise covariance matrix. We compared these upper bounds with the achievable sum-rate of the schemes in [20], [21]. We considered some consequences of the above results for the MIMO IC. Finally, we presented some numerical results and showed that the proposed sum-rate capacity upper bounds are tighter than existing bounds.

Appendix

We assume that the channel is used n times. The transmitted and received vector sequences are denoted by $\mathbf{x}_i^n$ and $\mathbf{y}_i^n$ for user $i = 1, 2$, and $\mathbf{x}_i^n$ satisfies the power constraint P_i. By Fano's inequality, we have for the MIMO ZC

$$H(W_{ii}|\mathbf{y}_i^n) \leq n\varepsilon_n, \qquad i = 1, 2,$$
$$H(W_{12}|\mathbf{y}_1^n) \leq n\varepsilon_n, \qquad (17)$$

where $\varepsilon_n \to 0$ as $n \to \infty$. We first introduce some lemmas which will be used in the proofs.

A. Preliminaries

Lemma 3: Let $\mathbf{y} = \mathbf{H}\,\mathbf{x} + \mathbf{n}$, where $\mathbf{H}$ is a $N \times M$ matrix, $N \geq M$ with full column rank, and $\mathbf{n} \sim \mathcal{CN}(0, \mathbf{I})$. Denote by $\mathbf{H}_l^{-1} = (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H$ the left inverse of $\mathbf{H}$, i.e., $\mathbf{H}_l^{-1}\mathbf{H} = \mathbf{I}$. Define $\mathbf{y}' \triangleq \mathbf{H}_l^{-1}\mathbf{y} = \mathbf{x} + \mathbf{n}'$, where $\mathbf{n}' \sim \mathcal{CN}(0, (\mathbf{H}^H\,\mathbf{H})^{-1})$. Then, $I(\mathbf{x};\mathbf{y}) = I(\mathbf{x};\mathbf{y}')$ implying that $\mathbf{y}'$ is a lossless representation of $\mathbf{y}$.

Proof: From the definition of $\mathbf{y}'$, we have the following Markov chain: $\mathbf{x} \to \mathbf{y} \to \mathbf{y}'$. Define $\tilde{\mathbf{y}} \triangleq \mathbf{H}\mathbf{y}' = \mathbf{H}\mathbf{x} + \mathbf{H}\mathbf{n}' = \mathbf{H}\mathbf{x} + \tilde{\mathbf{n}}$, where $\tilde{\mathbf{n}} \sim \mathcal{CN}(0, \mathbf{P_H})$, $\mathbf{P_H} = \mathbf{H}(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H$. We observe that $\mathbf{P_H}$ is in fact the unique orthogonal projection matrix onto the column space of $\mathbf{H}$. Consequently, it follows from the definition of a projection matrix that $\mathbf{P_H} \preceq \mathbf{I}$ [31], [39]. Thus, we can write $\mathbf{y} = \tilde{\mathbf{y}} + \hat{\mathbf{n}}$, where $\hat{\mathbf{n}} \sim \mathcal{CN}(0, \mathbf{I} - \mathbf{P_H})$. Using the above, we have the following inverse Markov chain $\mathbf{x} \to \mathbf{y}' \to \tilde{\mathbf{y}} \to \mathbf{y}$. From the above two Markov chains, we have $I(\mathbf{x};\mathbf{y}) = I(\mathbf{x};\mathbf{y}')$. ∎

Lemma 4 [30, Observation 7.7.2]: If $\mathbf{A} \succeq \mathbf{B}$ and $\mathbf{C}$ is any other matrix, then $\mathbf{CAC}^H \succeq \mathbf{CBC}^H$.

B. Proof of Statement (i) of Theorem 1

See Appendix A for some lemmas which are used in the proof. The proof is divided into two parts. In the first part of the proof, we transform the channel between transmitter 2 and receiver 2 into an equivalent channel. In the second part, we show that by appropriately reducing the noise at receiver 1 and using the zero-forcing (ZF) receive filter, receiver 1 can decode message $\mathbf{W}_{22}$ resulting in a MAC sum-rate upper bound.

Since a part of the proof is similar to the proofs of the sum-rate upper bounds for the MIMO ZC and the MIMO IC derived in [17], [40], respectively, we summarize the steps that are similar and highlight by giving additional details the steps that are different.

Since $N_2 \geq M_2$, by the assumption of full column rank, $\mathbf{H}_{22}^H\mathbf{H}_{22}$ is an invertible matrix. We use zero-forcing filter, denoted by $\mathbf{T}_{ZF} = (\mathbf{H}_{22}^H\mathbf{H}_{22})^{-1}\mathbf{H}_{22}^H$ at receiver 2 to get

$$\mathbf{y}_2' \triangleq \mathbf{T}_{ZF}\mathbf{y}_2 = \mathbf{x}_2 + \mathbf{T}_{ZF}\mathbf{n}_2 = \mathbf{x}_2 + \mathbf{n}_2', \qquad (18)$$

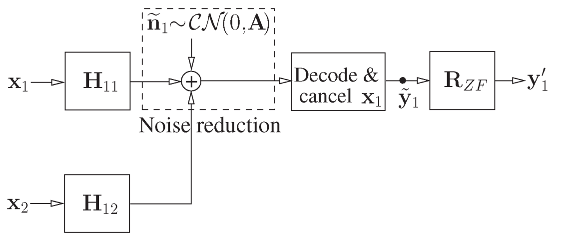

Fig. 7. Sequence of operations to show the decodability of W_{22} at receiver 1 in the MIMO ZC.

where $\mathbf{n}_2' \sim \mathcal{CN}[0, \mathbf{T}_{ZF}\mathbf{T}_{ZF}^H = (\mathbf{H}_{22}^H\mathbf{H}_{22})^{-1}]$. Using Lemma 3, $\mathbf{y}_2'$ is an equivalent representation of $\mathbf{y}_2$ and has resulted in dropping receive signal dimensions. These are not useful for decoding $\mathbf{x}_2$ at receiver 2, since they consist of only noise components.

Now, we use the following strategy to show the decodability of W_{22} at receiver 1. First, we reduce the noise appropriately at receiver 1 and then apply the zero forcing filter and show that receiver 1 can obtain a better channel to transmitter 2 than receiver 2. The sequence of operations employed at receiver 1 is illustrated in Fig. 7.

As shown in Fig. 3, $W_{11} \rightarrow \mathbf{x}_1^n$ is the only message emanating from transmitter 1. From (17), receiver 1 can decode message W_{11} with an arbitrarily low probability of error. Now, replace the noise $\mathbf{n}_1$ at receiver 1 with $\tilde{\mathbf{n}}_1$, where $\tilde{\mathbf{n}}_1 \sim \mathcal{CN}(0,\mathbf{A})$, with $\mathbf{A} \preceq \mathbf{I}$. We note that the reduction in the noise at receiver 1 does not hamper its ability to decode the intended messages W_{11}, W_{12}, since the original noise statistics can be obtained by adding appropriate noise. Let $\tilde{\mathbf{y}}_1$ denote the output at receiver 1 after $\mathbf{x}_1^n$ is decoded and canceled out, i.e., $\tilde{\mathbf{y}}_1 = \mathbf{H}_{12}\mathbf{x}_2 + \tilde{\mathbf{n}}_1$. Since $N_1 \geq M_2$, by the assumption of full column rank, $\mathbf{H}_{12}^H\mathbf{H}_{12}$ is an invertible matrix. We use zero-forcing filter, denoted by $\mathbf{R}_{ZF} = (\mathbf{H}_{12}^H\mathbf{H}_{12})^{-1}\mathbf{H}_{12}^H$ at receiver 1 to get

$$\mathbf{y}_1' \triangleq \mathbf{R}_{ZF}\tilde{\mathbf{y}}_1 = \mathbf{x}_2 + \mathbf{R}_{ZF}\tilde{\mathbf{n}}_1 = \mathbf{x}_2 + \mathbf{n}_1', \tag{19}$$

where $\mathbf{n}_1' \sim \mathcal{CN}(0, \mathbf{R}_{ZF}\mathbf{A}\mathbf{R}_{ZF}^H)$. Comparing (19) with (18), if $\mathbf{R}_{ZF}\mathbf{A}\mathbf{R}_{ZF}^H \preceq (\mathbf{H}_{22}^H\mathbf{H}_{22})^{-1}$, then we have,

$$I\left(\mathbf{x}_2^n;\mathbf{y}_1'^n\right) \geq I\left(\mathbf{x}_2^n;\mathbf{y}_2'^n\right) \stackrel{(a)}{=} I\left(\mathbf{x}_2^n;\mathbf{y}_2^n\right),$$

where the equality in (a) follows from Lemma 3. Therefore, we have

$$h\left(\mathbf{x}_2^n|\mathbf{y}_1'^n\right) \leq h\left(\mathbf{x}_2^n|\mathbf{y}_2'^n\right) = h\left(\mathbf{x}_2^n|\mathbf{y}_2^n\right).$$

Thus, receiver 1 can get a better channel to transmitter 2 than receiver 2. This in turn makes W_{22} decodable at receiver 1, which implies that receiver 1 can decode all three messages in the MIMO ZC. The sum-rate of the MAC formed by transmitters 1, 2 and receiver 1 is an upper bound on the sum-rate of the original MIMO ZC. Note that this MAC sum-rate upper bound is dependent on the noise covariance matrix $\mathbf{A}$ which must satisfy the two conditions: $\mathbf{A} \preceq \mathbf{I}$ and $\mathbf{R}_{ZF}\mathbf{A}\mathbf{R}_{ZF}^H \preceq (\mathbf{H}_{22}^H\mathbf{H}_{22})^{-1}$. Below, we derive a solution to the noise matrix $\mathbf{A}$, by making use of the concept of projection matrices [31].

We observe that multiplication of $\tilde{\mathbf{y}}_1$ by $\mathbf{R}_{ZF}$ to get $\mathbf{y}_1' \in \mathbb{C}^{M_2\times 1}$ in (19) has resulted in a reduction of the received signal dimension. As the name implies, the zero-forcing filter 'forces to zero' both signal and noise components that lie in the null space of $\mathbf{H}_{12}^H$, or equivalently those that are orthogonal to the column space of $\mathbf{H}_{12}$.

We denote by $\mathcal{W}$ the space spanned by the columns of $\mathbf{H}_{12}$ and the space orthogonal to $\mathcal{W}$ is denoted by $\mathcal{W}^{\perp}$. From the above observation, it is clear that, we need only reduce the noise components that lie in the subspace $\mathcal{W}$. Let $\mathbf{P}$ denote the orthogonal projection onto $\mathcal{W}$, $\mathbf{P}^2 = \mathbf{P}$, $\mathbf{P}^H = \mathbf{P}$. Since the columns of $\mathbf{H}_{12}$ form a basis for this subspace, we have $\mathbf{P} = \mathbf{H}_{12}(\mathbf{H}_{12}^H\mathbf{H}_{12})^{-1}\mathbf{H}_{12}^H$ and $\mathbf{I}-\mathbf{P}$ is the orthogonal projection onto $\mathcal{W}^{\perp}$ [39]. Any noise vector $\mathbf{n}_1 \in \mathbb{C}^{N_1\times 1}$ can be uniquely decomposed as $\mathbf{n}_1 = \mathbf{P}\mathbf{n}_1 + (\mathbf{I}-\mathbf{P})\mathbf{n}_1$ with $\mathbf{P}\mathbf{n}_1 \in \mathcal{W}$ and $(\mathbf{I}-\mathbf{P})\mathbf{n}_1 \in \mathcal{W}^{\perp}$. Note that $\mathbf{R}_{ZF}(\mathbf{I}-\mathbf{P})\mathbf{n}_1 = \mathbf{0}$. Using the above arguments, we decompose the reduced noise vector $\tilde{\mathbf{n}}_1$ as

$$\tilde{\mathbf{n}}_1 = (\mathbf{I}-\mathbf{P})\mathbf{n}_1 + \mathbf{w}_1,$$

where $\mathbf{w}_1 \in \mathcal{W}$. In the above decomposition, we have retained noise components of $\mathbf{n}_1$ that lie in $\mathcal{W}^{\perp}$, since they are forced to zero by the ZF filter. Since the columns of $\mathbf{H}_{12}$ form a basis for $\mathcal{W}$, we can write $\mathbf{w}_1 = \mathbf{H}_{12}\mathbf{w}_2$, where $\mathbf{w}_2 \in \mathbb{C}^{M_2\times 1}$, i.e., $\tilde{\mathbf{n}}_1 = (\mathbf{I}-\mathbf{P})\mathbf{n}_1 + \mathbf{H}_{12}\mathbf{w}_2$. By definition, we have

$$\begin{aligned}\mathbf{A} &= \mathbb{E}\left[\tilde{\mathbf{n}}_1\tilde{\mathbf{n}}_1^H\right] = \mathbb{E}\left[(\mathbf{I}-\mathbf{P})\mathbf{n}_1\mathbf{n}_1^H(\mathbf{I}-\mathbf{P})^H + \mathbf{H}_{12}\mathbf{w}_2\mathbf{w}_2^H\mathbf{H}_{12}^H\right]\\ &= (\mathbf{I}-\mathbf{P}) + \mathbf{H}_{12}\mathbf{W}\mathbf{H}_{12}^H.\end{aligned} \tag{20}$$

where $\mathbf{W} \triangleq \mathbb{E}[\mathbf{w}_2\mathbf{w}_2^H]$. Since we require $\mathbf{A} \preceq \mathbf{I}$, this translates to $\mathbf{H}_{12}\mathbf{W}\mathbf{H}_{12}^H \preceq \mathbf{P}$. From Lemma 4, we get the condition $\mathbf{W} \preceq (\mathbf{H}_{12}^H\mathbf{H}_{12})^{-1}$.

Using (20) and substituting for $\mathbf{A}$, the second condition $\mathbf{R}_{ZF}\mathbf{A}\mathbf{R}_{ZF}^H \preceq (\mathbf{H}_{22}^H\mathbf{H}_{22})^{-1}$ can be written as

$$\begin{aligned}&\mathbf{R}_{ZF}\left[(\mathbf{I}-\mathbf{P}) + \mathbf{H}_{12}\mathbf{W}\mathbf{H}_{12}^H\right]\mathbf{R}_{ZF}^H \preceq \left(\mathbf{H}_{22}^H\mathbf{H}_{22}\right)^{-1}\\ &\qquad\Rightarrow \mathbf{W} \stackrel{(a)}{\preceq} \left(\mathbf{H}_{22}^H\mathbf{H}_{22}\right)^{-1},\end{aligned}$$

where (a) follows from the fact that $\mathbf{R}_{ZF}(\mathbf{I}-\mathbf{P}) = \mathbf{0}$, and $\mathbf{R}_{ZF}\mathbf{H}_{12} = \mathbf{I}$. Thus, we have the two conditions $\mathbf{W} \preceq (\mathbf{H}_{12}^H\mathbf{H}_{12})^{-1}$ and $\mathbf{W} \preceq (\mathbf{H}_{22}^H\mathbf{H}_{22})^{-1}$. Note that for every choice of $\mathbf{W}$ satisfying the above conditions, we get an upper bound. The noise covariance matrix $\mathbf{A}$ is now given by $\mathbf{A} = \mathbf{I} - \mathbf{H}_{12}(\mathbf{H}_{12}^H\mathbf{H}_{12})^{-1}\mathbf{H}_{12}^H + \mathbf{H}_{12}\mathbf{W}\mathbf{H}_{12}^H$.

C. Primal-Dual Interior Point Algorithm

Consider the minimax problem formulation in (12). Let $\mathbf{C}_1 = \mathrm{diag}(\mathbf{I}_{N_1}, \mathbf{0}_{N_2\times N_2})$ and $\mathbf{C}_2 = \mathrm{diag}(\mathbf{0}_{N_1\times N_1}, \mathbf{I}_{N_2})$. The constraint that $\mathbf{Z} \in \mathbb{S}$ can be expressed as $\sum_{i=1}^2 \mathbf{C}_i\mathbf{Z}\mathbf{C}_i = \mathbf{I}_N$ with $\mathbf{Z} \succeq 0$. First, we write the objective function in (12) as $f_0(\mathbf{Z},\mathbf{S}_1,\mathbf{S}_2)$. Forming the Lagrangian for (12), we have

$$\begin{aligned}&f_0(\mathbf{Z},\mathbf{S}_1,\mathbf{S}_2) + \sum_{i=1}^{2}\lambda_i\left(\mathrm{Tr}(\mathbf{S}_i) - P_i\right) + \sum_{i=1}^{2}\mathrm{Tr}\left(\mathbf{\Phi}_i(-\mathbf{S}_i)\right)\\ &\qquad + \mathrm{Tr}\left(\mathbf{\Sigma}(\mathbf{C}_1\mathbf{Z}\mathbf{C}_1 + \mathbf{C}_2\mathbf{Z}\mathbf{C}_2 - \mathbf{I}_N)\right) + \mathrm{Tr}\left(\mathbf{\Gamma}(-\mathbf{Z})\right),\end{aligned}$$

where λ_i is the dual variable associated with the power constraint P_i, $i = 1, 2$. $\mathbf{\Phi}_i$, $\mathbf{\Gamma}$ are matrices of dual variables associated with the semidefinite constraint on $\mathbf{S}_i$, $\mathbf{Z}$, respectively, and $\boldsymbol{\Sigma}$ is a block diagonal matrix of dual variables associated with the equality constraint on $\mathbf{Z}$. The starting point for the derivation of the interior point method is the modified KKT conditions for (12). For $i = 1, 2$, we have

$$\mathbf{Z}^* \succeq 0,\ \mathbf{S}_i^* \succeq 0,\ \lambda_i^* \geq 0,\ \boldsymbol{\Gamma}^* \succeq 0,\ \boldsymbol{\Phi}_i^* \succeq 0, \tag{21}$$

$$\mathbf{C}_1\mathbf{Z}^*\mathbf{C}_1 + \mathbf{C}_2\mathbf{Z}^*\mathbf{C}_2 = \mathbf{I}_N, \tag{22}$$

$$\mathbf{Z}^*\boldsymbol{\Gamma}^* = \boldsymbol{\Gamma}^*\mathbf{Z}^* = 0,\ \mathbf{S}_i^*\boldsymbol{\Phi}_i^* = \boldsymbol{\Phi}_i^*\mathbf{S}_i^* = 0, \tag{23}$$

$$-\lambda_i^*\left(\mathrm{Tr}\left(\mathbf{S}_i^*\right) - P_i\right) = 1/a, \tag{24}$$

$$\mathbf{C}_1\boldsymbol{\Sigma}^*\mathbf{C}_2 + \mathbf{C}_2\boldsymbol{\Sigma}^*\mathbf{C}_1 = 0, \tag{25}$$

$$-\nabla_{\mathbf{S}_i} f_0\left(\mathbf{Z}, \mathbf{S}_i^*, \mathbf{S}_j\right) + \lambda_i^*\mathbf{I} - \boldsymbol{\Phi}_i^{*T} = 0,\ i \neq j, \tag{26}$$

$$\nabla_{\mathbf{Z}} f_0(\mathbf{Z}^*, \mathbf{S}_1, \mathbf{S}_2) + \sum_{i=1}^{2} \mathbf{C}_i\boldsymbol{\Sigma}^{*T}\mathbf{C}_i - \boldsymbol{\Gamma}^{*T} = 0, \tag{27}$$

where $a > 0$ and $(\cdot)^*$ indicates the optimal value of the variable at the saddle point. Since strong duality holds in case of (12), (23)–(24) represent the modified complementary slackness conditions. To find the saddle point, we need to simultaneously solve the system of (21)–(27). For $i = 1, 2$, let

$$\mathbf{Z}^* = \mathbf{Z} + \Delta\mathbf{Z}, \quad \mathbf{S}_i^* = \mathbf{S}_i + \Delta\mathbf{S}_i, \quad \lambda_i^* = \lambda_i + \Delta\lambda_i, \tag{28}$$

$$\boldsymbol{\Gamma}^* = \boldsymbol{\Gamma} + \Delta\boldsymbol{\Gamma}, \quad \boldsymbol{\Phi}_i^* = \boldsymbol{\Phi}_i + \Delta\boldsymbol{\Phi}_i, \quad \boldsymbol{\Sigma}^* = \boldsymbol{\Sigma} + \Delta\boldsymbol{\Sigma}, \tag{29}$$

where $\Delta\mathbf{S}_i, \Delta\mathbf{Z}$ are the primal search directions, and $\Delta\lambda_i$, $\Delta\boldsymbol{\Phi}_i$, $\Delta\boldsymbol{\Gamma}$ and $\Delta\boldsymbol{\Sigma}$ are the dual search directions.

We describe the algorithm used to solve (12) in Algorithm 1. Let $\mathbf{x_Z} = \mathrm{vec}([\mathbf{Z}\ \boldsymbol{\Gamma}\ \boldsymbol{\Sigma}])$ and $\mathbf{x}_{\mathbf{S}_i} = \mathrm{vec}([\mathbf{S}_i\ \boldsymbol{\Phi}_i\ \lambda_i])$, $i = 1, 2$. μ, ε are parameters of the algorithm and m denotes the number of modified complementary slackness conditions, where $m = 2$ from (24). In the interest of brevity, we refrain from going into the details of the algorithm. See [37, Section 11.7] for a detailed exposition of the different nuances associated with the interior point algorithm. The details of the algorithm specific to the minimax problem in (12) are given in [29].

The updated primal and dual variables in the kth iteration of the algorithm do not satisfy the KKT conditions (21)–(27), except in the limit as the algorithm converges. Hence, we define the primal and dual residuals w.r.t $\mathbf{Z}$ and $\mathbf{S}_i$, $i = 1, 2$ at the kth iteration as:

$$\begin{aligned}
\mathbf{R}_{\mathbf{S}_i}^{dual} &= \nabla_{\mathbf{S}_i} f_0 - \lambda_i\mathbf{I}_{M_i} + \boldsymbol{\Phi}_i^T, \quad i = 1, 2, \\
\mathbf{R}_{\mathbf{Z}}^{dual} &= \nabla_{\mathbf{Z}} f_0 + (\boldsymbol{\Sigma} - \boldsymbol{\Gamma})^T, \\
\mathbf{R}_{\mathbf{Z}}^{pri} &= \mathbf{C}_1\mathbf{Z}\mathbf{C}_1 + \mathbf{C}_2\mathbf{Z}\mathbf{C}_2 - \mathbf{I}_N.
\end{aligned} \tag{30}$$

The primal and dual search directions, $\Delta\mathbf{x_Z}$, $\Delta\mathbf{x}_{\mathbf{S}_i}$ can be computed by substituting (28), (29) in (21)–(27), and using the first-order Taylor's approximation for the resulting system of matrix equations. The line search in Algorithm 1 is a standard backtracking line search, based on the norm of the primal and dual residuals, modified to ensure that $\mathbf{Z} \succ 0$, $\mathbf{S}_i \succ 0$, $\boldsymbol{\Phi}_i \succ 0$, $\boldsymbol{\Gamma} \succ 0$ and $\lambda_i > 0$ for $i = 1, 2$ [37]. The parameter $\hat{\eta}$ is called the surrogate duality gap. This would be the duality gap if the primal and the dual residuals in (30) were equal to zero. It is given by

$$\hat{\boldsymbol{\eta}} = \sum_{i=1}^{2} \left[\lambda_i\left(P_i - \mathrm{Tr}(\mathbf{S}_i)\right) + \mathrm{Tr}\left(\boldsymbol{\Phi}_i\mathbf{S}_i\mathbf{S}_i^T\boldsymbol{\Phi}_i^T\right)\right] + \mathrm{Tr}(\boldsymbol{\Gamma}\mathbf{Z}\mathbf{Z}^T\boldsymbol{\Gamma}^T).$$

The convergence of Algorithm 1 follows from the convergence of the primal-dual interior point method [37]. Lastly, we observed that values of the parameter μ on the order of 10 resulted in faster convergence.

Algorithm 1 Primal-dual interior point method

1) *Initialize* $\mathbf{Z} \succ 0$, $\boldsymbol{\Gamma} \succ 0$, $\boldsymbol{\Sigma} = 0$, $\mathbf{S}_i \succ 0$, $\boldsymbol{\Phi}_i \succ 0$, $\lambda_i > 0$, $i = 1, 2$, $\mu > 1$, $\varepsilon > 0$.
2) *Evaluate* $a = \mu m/\hat{\eta}$.
3) *Compute primal-dual search directions*, $\Delta\mathbf{x_Z}$, $\Delta\mathbf{x}_{\mathbf{S}_i}$, $i = 1, 2$.
4) *Line search and update*: Determine step length $v > 0$, $u_i > 0$ and set $\mathbf{x_Z} = \mathbf{x_Z} + v\Delta\mathbf{x_Z}$, $\mathbf{x}_{\mathbf{S}_i} = \mathbf{x}_{\mathbf{S}_i} + u_i\Delta\mathbf{x}_{\mathbf{S}_i}$.
5) *Compute primal and dual residuals*: $\mathbf{R}_{\mathbf{S}_i}^{dual}$, $\mathbf{R}_{\mathbf{Z}}^{pri}$, $\mathbf{R}_{\mathbf{Z}}^{dual}$ and *surrogate duality gap* $\hat{\eta}$.
6) *If* $\|\mathbf{R}_{\mathbf{S}_i}^{dual}\|_F \leq \varepsilon$, $\|\mathbf{R}_{\mathbf{Z}}^{pri}\|_F \leq \varepsilon$, $\|\mathbf{R}_{\mathbf{Z}}^{dual}\|_F \leq \varepsilon$ and $\hat{\eta} \leq \varepsilon$, *stop. Otherwise go to step 2.*

REFERENCES

[1] T. Han and K. Kobayashi, "A new achievable rate region for the interference channel," *IEEE Trans. Inf. Theory*, vol. IT-27, no. 1, pp. 49–60, Jan. 1981.
[2] G. Kramer, "Outer bounds on the capacity of Gaussian interference channels," *IEEE Trans. Inf. Theory*, vol. 50, no. 3, pp. 581–586, Mar. 2004.
[3] R. H. Etkin, D. N. C. Tse, and H. Wang, "Gaussian interference channel capacity to within one bit," *IEEE Trans. Inf. Theory*, vol. 54, no. 12, pp. 5534–5562, Dec. 2008.
[4] X. Shang, G. Kramer, and B. Chen, "A new outer bound and the noisy-interference sum-rate capacity for the Gaussian interference channels," *IEEE Trans. Inf. Theory*, vol. 55, no. 2, pp. 689–699, Feb. 2009.
[5] A. S. Motahari and A. K. Khandani, "Capacity bounds for the Gaussian interference channel," *IEEE Trans. Inf. Theory*, vol. 55, no. 2, pp. 620–643, Feb. 2009.
[6] V. S. Annapureddy and V. Veeravalli, "Gaussian interference networks: Sum capacity in the low-interference regime and new outer bounds on the capacity region," *IEEE Trans. Inf. Theory*, vol. 55, no. 7, pp. 3032–3050, Jul. 2009.
[7] N. Liu and S. Ulukus, "On the capacity region of the Gaussian Z-channel," in *Proc. IEEE GLOBECOM*, Dallas, TX, USA, Dec. 2004, vol. 1, pp. 415–419.
[8] H. F. Chong, M. Motani, and H. K. Garg, "Capacity theorems for the "Z" channel," *IEEE Trans. Inf. Theory*, vol. 53, no. 4, pp. 1348–1365, Apr. 2007.
[9] O. Koyluoglu, M. Shahmohammadi, and H. El Gamal, "A new achievable rate region for the discrete memoryless X channel," in *Proc. IEEE ISIT*, Seoul, Korea, Jul. 2009, pp. 2427–2431.
[10] C. Huang, S. A. Jafar, and V. R. Cadambe, "Interference alignment and the generalized degrees of freedom of the X channel," *IEEE Trans. Inf. Theory*, vol. 58, no. 8, pp. 5130–5150, Aug. 2012.
[11] Q. Li *et al.*, "MIMO techniques in WiMAX and LTE: A feature overview," *IEEE Commun. Mag.*, vol. 48, no. 5, pp. 86–92, May 2010.
[12] X. Shang, B. Chen, G. Kramer, and H. V. Poor, "Capacity regions and sum-rate capacities of vector Gaussian interference channels," *IEEE Trans. Inf. Theory*, vol. 56, no. 10, pp. 5030–5044, Oct. 2010.
[13] V. S. Annapureddy and V. V. Veeravalli, "Sum capacity of MIMO interference channels in the low interference regime," *IEEE Trans. Inf. Theory*, vol. 57, no. 5, pp. 2565–2581, May 2011.
[14] X. Shang and H. V. Poor, "Noisy-interference sum-rate capacity for vector Gaussian interference channels," *IEEE Trans. Inf. Theory*, vol. 59, no. 1, pp. 132–153, Jan. 2013.
[15] P. Mohapatra and C. R. Murthy, "Outer bounds on the sum rate of the K-user MIMO Gaussian interference channel," *IEEE Trans. Commun.*, vol. 61, no. 1, pp. 176–186, Jan. 2013.
[16] S. Karmakar and M. K. Varanasi, "The capacity region of the MIMO interference channel and its reciprocity to within a constant gap," *IEEE Trans. Inf. Theory*, vol. 59, no. 8, pp. 4781–4797, Aug. 2013.
[17] S. A. Jafar and S. Shamai (Shitz), "Degrees of freedom region of the MIMO X channel," *IEEE Trans. Inf. Theory*, vol. 54, no. 1, pp. 151–170, Jan. 2008.

[18] M. A. Maddah-Ali, A. S. Motahari, and A. K. Khandani, "Communication over X channel: Signalling and multiplexing gain," Univ. Waterloo, Waterloo, ON, Canada, Tech. Rep. UW-ECE-2006-12, 2006.
[19] S. A. Jafar, "Degrees of freedom on the MIMO X channel—The optimality of the MMK scheme," arXiv:cs 0607099v1, Jul. 2006.
[20] M. A. Maddah-Ali, A. S. Motahari, and A. K. Khandani, "Communication over MIMO X channels: Interference alignment, decomposition, performance analysis," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3457–3470, Aug. 2008.
[21] M. A. Maddah-Ali, A. S. Motahari, and A. K. Khandani, "Signaling over MIMO multi-base systems: Combination of multi-access and broadcast schemes," in *Proc. IEEE ISIT*, Seattle, WA, USA, Jul. 2006, pp. 2104–2108.
[22] A. Dotzler, D. Schmidt, G. Dietl, and W. Utschick, "Gradient projection based interference alignment for the MIMO X channel," in *Proc. Int. WSA*, Bremen, Germany, Feb. 2010, pp. 355–360.
[23] A. Agustin and J. Vidal, "Improved interference alignment precoding for the MIMO X channel," in *Proc. IEEE ICC*, Kyoto, Japan, Jun. 2011, pp. 1–5.
[24] S. Park and Y. Ko, "*K*-user MIMO X network system with perfect interference alignment," in *Proc. IEEE ICC*, Kyoto, Japan, Jun. 2011, pp. 1–5.
[25] F. Li and H. Jafarkhani, "Space-time processing for X channels using precoders," *IEEE Trans. Signal Process.*, vol. 60, no. 4, pp. 1849–1861, Apr. 2012.
[26] P. Kumar and S. Bhashyam, "MIMO Gaussian X channel: Noisy interference regime," *IEEE Commun. Lett.*, vol. 18, no. 8, pp. 1295–1298, Aug. 2014.
[27] U. Niesen and M. Maddah-Ali, "Interference alignment: From degrees of freedom to constant-gap capacity approximations," *IEEE Trans. Inf. Theory*, vol. 59, no. 8, pp. 4855–4888, Aug. 2013.
[28] A. S. Avestimehr, S. N. Diggavi, and D. N. C. Tse, "Wireless network information flow: A deterministic approach," *IEEE Trans. Inf. Theory*, vol. 57, no. 4, pp. 1872–1905, Apr. 2011.
[29] R. Prasad, N. Srinidhi, and A. Chockalingam, "Bounds on the sum-rate capacity of the Gaussian MIMO X channel," in *Proc. 50th Annu. Allerton Conf. Commun., Control, Comput.*, Monticello, IL, USA, Oct. 2012, pp. 1238–1245.
[30] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1985.
[31] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed. Baltimore, MD, USA: The Johns Hopkins Univ. Press, 1996.
[32] S. Vishwanath, N. Jindal, and A. Goldsmith, "Duality, achievable rates, sum-rate capacity of MIMO broadcast channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2895–2909, Oct. 2003.
[33] P. Viswanath and D. Tse, "Sum capacity of the vector Gaussian broadcast channel and uplink-downlink duality," *IEEE Trans. Inf. Theory*, vol. 49, no. 8, pp. 1912–1921, Aug. 2003.
[34] W. Yu, "Uplink-downlink duality via minimax duality," *IEEE Trans. Inf. Theory*, vol. 52, no. 2, pp. 361–374, Feb. 2006.
[35] S. Vishwanath, S. Boyd, and A. Goldsmith, "Worst-case capacity of Gaussian vector channels," in *Proc. Can. Workshop Inf. Theory*, Waterloo, ON, Canada, May 2003, pp. 1–5.
[36] *CVX: Matlab Software for Disciplined Convex Programming*, CVX Research and Inc., Austin, TX, USA, Sep. 2012. [Online]. Available: http://cvxr.com/cvx
[37] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2003.
[38] W. Yu, W. Rhee, S. Boyd, and J. M. Cioffi, "Iterative water-filling for Gaussian vector multiple-access channels," *IEEE Trans. Inf. Theory*, vol. 50, no. 1, pp. 145–152, Jan. 2004.
[39] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*. Philadelphia, PA, USA: SIAM, 2000.
[40] S. A. Jafar and M. Fakhereddin, "Degrees of freedom for the MIMO interference channel," *IEEE Trans. Inf. Theory*, vol. 53, no. 7, pp. 145–152, Jan. 2007.

Ranga Prasad received the B.E. degree in electronics and communication engineering from the University Visvesvaraya College of Engineering (UVCE), Bangalore, India, in 2005. He is pursuing the Ph.D. degree at the Department of Electrical Communication Engineering, Indian Institute of Science (IISc), Bangalore, India. He worked as an Engineer at Ittiam Systems Private Limited, Bangalore, India, from October 2005 to July 2008, on the implementation of video codecs such as MPEG4 and WMV9 on multi-core embedded platforms. His research interests include information theoretic limits of wireless communications and the applications of information theory to emerging areas in science, economics, and engineering.

Srikrishna Bhashyam (S'96–M'02–SM'08) received the B.Tech. degree in electronics and communication engineering from the Indian Institute of Technology, Madras, India, in 1996 and the M.S. and Ph.D. degrees in electrical and computer engineering from Rice University, Houston, TX, USA, in 1998 and 2001, respectively. He worked as a Senior Engineer at Qualcomm, Inc., Campbell, CA, USA from June 2001 to March 2003, on wideband code-division multiple access (WCDMA) modem design. Since May 2003, he is at the Indian Institute of Technology, Madras. He is now an Associate Professor in the Department of Electrical Engineering. He served as an Editor of IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS during 2009–2014. His research interests are in communication and information theory, wireless networks, and statistical signal processing.

A. Chockalingam (S'92–M'93–SM'98) was born in Rajapalayam, Tamil Nadu, India. He received the B.E. (Honors) degree in electronics and communication engineering from the P.S.G. College of Technology, Coimbatore, India, in 1984, the M.Tech. degree in electronics and electrical communications engineering (with specialization in satellite communications) from the Indian Institute of Technology, Kharagpur, India, in 1985, and the Ph.D. degree in electrical communication engineering (ECE) from the Indian Institute of Science (IISc), Bangalore, India, in 1993. During 1986 to 1993, he worked with the Transmission R&D Division of the Indian Telephone Industries Limited, Bangalore. From December 1993 to May 1996, he was a Postdoctoral Fellow and an Assistant Project Scientist at the Department of Electrical and Computer Engineering, University of California, San Diego. From May 1996 to December 1998, he served Qualcomm, Inc., San Diego, CA, as a Staff Engineer/Manager in the systems engineering group. In December 1998, he joined the faculty of the Department of ECE, IISc, Bangalore, India, where he is a Professor, working in the area of wireless communications and networking.

Dr. Chockalingam is a recipient of the Swarnajayanti Fellowship from the Department of Science and Technology, Government of India. He served as an Associate Editor of the IEEE TRANSACTIONS ON VEHICULAR TECHNOLOGY, and as an Editor of the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS. He served as a Guest Editor for the IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS (Special Issue on Multiuser Detection for Advanced Communication Systems and Networks), and for the IEEE JOURNAL OF SELECTED TOPICS IN SIGNAL PROCESSING (Special Issue on Soft Detection on Wireless Transmission). He is a Fellow of the Indian National Academy of Engineering, the National Academy of Sciences, India, and the Indian National Science Academy.