On the Sum Rate of a 2 × 2 Interference Network

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***Abstract*—In an $M \times N$ interference network, there are M transmitters and N receivers with each transmitter having independent messages for each of the $2^N - 1$ possible non-empty subsets of the receivers. We consider the 2 × 2 interference network with 6 possible messages, of which the 2 × 2 interference channel and X channel are special cases obtained by using only 2 and 4 messages respectively. Starting from an achievable rate region similar to the Han-Kobayashi region, we obtain an achievable sum rate. For the Gaussian interference network, we determine which of the 6 messages are sufficient for maximizing the sum rate within this rate region for the low, mixed, and strong interference conditions. It is observed that 2 messages are sufficient in several cases. Finally, we show that sum capacity is achieved using only 2 messages for a subset of the mixed interference conditions.**

I. INTRODUCTION

The Interference Network (IN) was introduced by Carleial [1] as a multi-terminal communication problem involving M transmitters and N receivers with each transmitter having independent messages for each of the $2^N - 1$ possible non-empty subsets of the receivers. Thus, a total of $M(2^N - 1)$ messages are transmitted across the channel leading to a $M(2^N - 1)$ dimensional capacity region. The multiple access channel (MAC), broadcast channel (BC), interference channel (IC), and X channel are all special cases of the Interference network (IN). For example, when $M = N$ and transmitter k is interested in communication with only receiver k, we have the M user IC. In the two user X channel, each transmitter $Tx_i, i \in \{1, 2\}$ has 2 *independent* messages corresponding to the two receivers, i.e., four messages in total.

The IC has been studied extensively in [1–8]. While the capacity region is unknown, several inner and outer bounds have been derived for the capacity region and the sum capacity [3, 5–8]. Under some channel conditions (or interference conditions), capacity or sum capacity has been determined [1–4]. The X channel has been studied in [9–14] to obtain capacity region bounds and generalized degrees of freedom.

Using all $M(2^N - 1)$ messages has been observed to be important when interference networks arise as states in a half-duplex relay network [15, 16]. In half-duplex relay networks, the set of transmitters and receivers at any given time instant form an interference network. The choice of rates for the $M(2^N - 1)$ messages depends on the overall information flow constraints. Therefore, a characterization of the $M(2^N - 1)$ dimensional rate region is useful in flow optimization. The messages that result in optimal flow will depend on the connectivity and the channel conditions of the links. In the context of X channels, it has been seen that using 2 messages is sum rate optimal under a subset of low and strong interference conditions [9, 17]. We consider the more general IN and determine which of the 6 messages are useful for all interference conditions.

The achievable rate region obtained using Han-Kobayashi type public-private message splitting of the 4 messages on the X channel in [14] provides an achievable rate region for the 2 × 2 IN. In this paper, we obtain the following results: (1) Starting from an achievable rate region in [14], we first obtain an achievable sum rate of the 2 × 2 IN. (2) For the Gaussian interference network, we determine which of the 6 messages are sufficient for maximizing the sum rate within this rate region for the low, mixed, and strong interference conditions. It is observed that 2 messages are sufficient in several cases. (3) Finally, we show that sum capacity is achieved using only 2 messages for a subset of the mixed interference conditions.

II. TWO USER DISCRETE MEMORYLESS IN (DMIN)

The 2 × 2 DMIN shown in Fig. 1 is a communication model where there are 3 messages from each transmitter. The messages from Tx_1 are:

1) Direct private message U_1 to Rx_1.
2) Common message V_1 to both receivers $\{Rx_1, Rx_2\}$.
3) Cross private message W_1 to Rx_2.

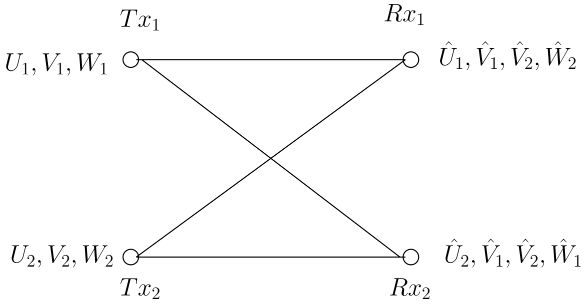

Fig. 1. 2 × 2 Interference Network

Similarly the messages U_2, V_2 and W_2 originate from Tx_2 communicating with Rx_2, $\{Rx_1, Rx_2\}$ and Rx_1 respectively. The receiver Rx_1 will decode 4 messages namely U_1, V_1, V_2 and W_2. Similarly, Rx_2 will decode U_2, V_1, V_2 and W_1.

Although an achievable rate region for the two user DMIN has not been explicitly reported, we can see that Han-Kobayashi (HK) [3] type message splitting on the X channel, given in [14], addresses the same problem as the IN. The HK [3] scheme, originally proposed for two user IC, allows *partial* decoding of *interference* at the unintended receiver so that a *common* part of the interference can be decoded (and subtracted) leading to better reception of its intended

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signal. The intended receiver decodes a private message, which cannot be decoded at the other receiver, and also decodes *this* common message combining them to form its *total* message. In [14], HK message splitting is applied to each of the 4 messages of the X channel leading to 8 (4×2) messages and an achievable region is given. It is easy to see that 2 *public* messages originating from each transmitter can be clubbed together as a single *public* message, resulting in a total of 6 messages. Here, we present an achievable rate region below for the 6 IN messages.

Let $Z = QU_1V_1W_1X_1U_2V_2W_2X_2Y_1Y_2 \in \Omega$,where Ω is the set of all probability distributions over the variables. $U_1, V_1, W_1, U_2, V_2, W_2$ are auxiliary random variables and X_1, X_2, Y_1, Y_2 are random variables on $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ respectively satisfying:

1) $U_1, V_1, W_1, U_2, V_2, W_2$ are mutually independent given Q,the time sharing random variable.
2) $X_1 = f_1(U_1, V_1, W_1|Q)$, $X_2 = f_2(U_2, V_2, W_2|Q)$, where f_1 and f_2 are deterministic functions of their arguments.

Let $\mathcal{R}(Z)$ denote the rate region formed by the six tuple rate $(R_{U_1}, R_{V_1}, R_{W_1}, R_{U_2}, R_{V_2}, R_{W_2})$ satisfying the following constraints:

$$R_{S_1} = \sum_{s \in S_1} R_s \le I(S_1; Y_1|\bar{S}_1, Q) \quad \forall S_1, \tag{1}$$

where S_1 is any non-empty subset of $M_1 = \{U_1, V_1, V_2, W_2\}$, and $\bar{S}_1 = M_1 \backslash S_1$. Since there are 15 possible subsets S_1, we have 15 constraints. Similarly, considering Rx_2, we get another 15 constraints corresponding to each non-empty subset S_2 of $M_2 = \{U_2, V_1, V_2, W_1\}$. For example, one of 30 constraints is $R_{U_2} + R_{V_2} + R_{W_1} \le I(U_2, V_2, W_1; Y_2|V_1, Q)$.

Let $\mathcal{R}_{IN}$ be the closure of $\bigcup_{Z \in \Omega} \mathcal{R}(Z)$. Then, any rate tuple in $\mathcal{R}_{IN}$ is achievable for the two user DMIN. The proof of achievability uses jointly typical decoding and is similar to the proof in [3].

III. ACHIEVABLE SUM RATE

Let the sum rate $S = R_{U_1} + R_{V_1} + R_{W_1} + R_{U_2} + R_{V_2} + R_{W_2}$.

Theorem 1: The achievable sum rate S is bounded as follows.

$$S \le \min\{T_1, T_2, T_3, T_4\}, \tag{2}$$

where

$$\begin{aligned} T_1 &= I(U_2, V_1, V_2, W_1; Y_2|Q) + I(U_1, W_2; Y_1|V_1, V_2, Q), \\ T_2 &= I(U_1, V_1, V_2, W_2; Y_1|Q) + I(U_2, W_1; Y_2|V_1, V_2, Q), \\ T_3 &= I(U_1, V_2, W_2; Y_1|V_1, Q) + I(U_2, V_1, W_1; Y_2|V_2, Q). \\ T_4 &= I(U_1, V_1, W_2; Y_1|V_2, Q) + I(U_2, V_2, W_1; Y_2|V_1, Q), \end{aligned}$$

Proof: See Appendix A. ■

It is worth noting that the sum rate can be bounded by several expressions using the constraints in (1). For example, by adding the bounds on $R_{U_1} + R_{V_1} + R_{V_2}$, $R_{U_2} + R_{W_1}$ and R_{W_2} in three of the constraints, we can get a bound for S. One can obtain similar bounds by several groupings of the 6 rate components of sum rate and adding the corresponding constraints from the rate region. The sum rate is bounded by the minimum of all such bounds. In the proof, it is shown that show that only 4 of the combinations are useful and the others are redundant.

IV. GAUSSIAN INTERFERENCE NETWORK (GIN)

The standard form for the Gaussian IN [1] is

$$\begin{aligned} Y_1 &= X_1 + h_2X_2 + Z_1 \\ Y_2 &= X_2 + h_1X_1 + Z_2 \end{aligned} \tag{3}$$

where $Z_1, Z_2 \sim N(0, 1)$. Power constraint P_1, P_2 are imposed on Tx_1 and Tx_2 respectively. The channel (or interference) conditions for the two user GIN can be classified into the following cases.

1) *Low Interference (LI)*: $0 \le h_1 \le 1, 0 \le h_2 \le 1$.
2) *Mixed Interference (MI)*: $0 \le h_1 \le 1, h_2 \ge 1$ or $0 \le h_2 \le 1, h_1 \ge 1$.
3) *Strong Interference (SI)*: $1 \le h_1^2 \le P_2 + 1, 1 \le h_2^2 \le P_1 + 1$.
4) *Very Strong Interference (VSI)*: $h_1^2 \ge P_2 + 1, h_2^2 \ge P_1 + 1$.

For the GIN, the DMIN rate region can be extended as follows. We consider the non-time sharing case, where $Q = \phi$, a constant. Further, we limit ourselves to $X_1 = U_1 + V_1 + W_1, X_2 = U_2 + V_2 + W_2$, where $U_1, V_1, W_1, U_2, V_2, W_2$ are independent Gaussian codebooks. Let us denote this rate region as $\mathcal{R}_{GIN}$. We employ superposition coding [18] at the transmitters with power distribution defined as follows. Messages U_i, V_i, and W_i are transmitted using powers α_iP_i, β_iP_i, and γ_iP_i, respectively, for $i = 1, 2$. Also, $\alpha_i + \beta_i + \gamma_i = 1$. Let $I_1 = 1 + h_2^2\alpha_2P_2 + \gamma_1P_1$ and $I_2 = 1 + h_1^2\alpha_1P_1 + \gamma_2P_2$. Let $C(x) = 0.5\log_2(1 + x)$. An achievable rate region for the two user GIN is once again defined by 30 constraints. The 15 constraints corresponding equation (1) are given by:

$$R_{S_1} = \sum_{s \in S_1} R_s \quad \le \quad C\left(\frac{\sum_s P_s}{I_1}\right),$$

where $P_{U_1} = \alpha_1P_1$, $P_{V_1} = \beta_1P_1$, $P_{V_2} = h_2^2\beta_2P_2$, and $P_{W_2} = h_2^2\gamma_2P_2$. Similarly, 15 more constraints can be written considering the rate constraints for Rx_2. Note that

$$\mathcal{R}_{GIN} \subseteq \mathcal{R}_{GIN_Q} \subseteq \mathcal{R}_{IN} \subseteq C_{IN},$$

where $\mathcal{R}_{GIN_Q}$ is the rate-region with optimal time sharing $(Q \ne \phi)$ strategy and Gaussian input. $\mathcal{R}_{IN}$ is the optimal time sharing strategy with optimal input distribution and C_{IN} is the capacity of the IN.

V. ACHIEVABLE SUM RATES IN GIN

In this section, we determine which of the 6 messages in the IN are useful in maximizing the sum rate for the various interference conditions. In [9], a similar question was answered for the X channel in a subset of the low and strong interference regimes extending the result for IC in [2]. Since the sum capacity of an IN is unknown, we first study the maximum sum rate within the achievable rate region described in the previous section as summarized in Table I. In Section VI, we show that the sum capacity is indeed achieved for some mixed interference conditions.

TABLE I
SUMMARY OF RESULTS

Region	Sub-region	Message-set
L.I	-	U_1, V_1, U_2, V_2
	$\lvert h_1(1+h_2^2P_2)+h_2(1+h_1^2P_1)\rvert \leq 1$	U_1, U_2
M.I	$0 \leq h_1 \leq 1, h_2 \geq 1$	U_1, W_2
	$0 \leq h_2 \leq 1, h_1 \geq 1$	U_2, W_1
S.I	-	W_1, V_1, W_2, V_2
V.S.I	-	W_1, V_1, W_2, V_2
	$\lvert h_1^{-1}(1+P_2)+h_2^{-1}(1+P_1)\rvert \leq 1$	W_1, W_2

For the GIN, the terms T_1, T_2, T_3, and T_4 in the sum rate bound in equation (2) are:

$$T_1 = C\left(\frac{\alpha_1P_1 + h_2^2\gamma_2P_2}{1+h_2^2\alpha_2P_2+\gamma_1P_1}\right) + C\left(\frac{\bar{\gamma}_2P_2 + h_1^2\bar{\alpha}_1P_1}{1+h_1^2\alpha_1P_1+\gamma_2P_2}\right),$$
$$T_2 = C\left(\frac{\bar{\gamma}_1P_1 + h_2^2\bar{\alpha}_2P_2}{1+h_2^2\alpha_2P_2+\gamma_1P_1}\right) + C\left(\frac{\alpha_2P_2 + h_1^2\gamma_1P_1}{1+h_1^2\alpha_1P_1+\gamma_2P_2}\right),$$
$$T_3 = C\left(\frac{\alpha_1P_1 + h_2^2\bar{\alpha}_2P_2}{1+h_2^2\alpha_2P_2+\gamma_1P_1}\right) + C\left(\frac{\alpha_2P_2 + h_1^2\bar{\alpha}_1P_1}{1+h_1^2\alpha_1P_1+\gamma_2P_2}\right),$$
$$T_4 = C\left(\frac{\bar{\gamma}_1P_1 + h_2^2\gamma_2P_2}{1+h_2^2\alpha_2P_2+\gamma_1P_1}\right) + C\left(\frac{\bar{\gamma}_2P_2 + h_1^2\gamma_1P_1}{1+h_1^2\alpha_1P_1+\gamma_2P_2}\right),$$

where $\bar{\alpha}_i = 1-\alpha_i$ and $\bar{\gamma}_i = 1-\gamma_i$.

A. Mixed Interference

There are two cases for Mixed Interference: (i) $0 \leq h_2 \leq 1, h_1 \geq 1$, and (ii) $0 \leq h_1 \leq 1, h_2 \geq 1$.

Theorem 2:
1) For case (i), the achievable sum rate is maximized by transmitting only U_2 and W_1, both to Rx_2. The sum rate achieved is the MAC sum capacity at $Rx_2 = C(h_1^2P_1 + P_2)$.
2) For case (ii), the achievable sum rate is maximized by transmitting only U_1 and W_2, both to Rx_1. The sum rate achieved is the MAC sum capacity at $Rx_1 = C(h_2^2P_2 + P_1)$.

Proof: The proof of statement (1) is in Appendix B. The other statement can be proved similarly by swapping indices 1 and 2. ■

B. Low Interference

Theorem 3:
1) Let $T_i = t_i$ for $i = 1, 2, 3, 4$ when the power sharing fractions are α_1, β_1, γ_1, α_2, β_2, γ_2. Let $T_i = t_i^{'}$ when the power sharing fractions are $\alpha_1^{'} = \alpha_1$, $\beta_1^{'} = \beta_1 + \gamma_1$, $\gamma_1^{'} = 0$, $\alpha_2^{'} = \alpha_2$, $\beta_2^{'} = \beta_2 + \gamma_2$, $\gamma_2^{'} = 0$. Then $t_i^{'} \geq t_i$ for $i = 1, 2, 3, 4$ if $0 \leq h_1, h_2 \leq 1$.
2) Messages W_1 and W_2 are not required to maximize the sum rate when $0 \leq h_1, h_2 \leq 1$.

Proof: See Appendix C. ■

From the theorem above, it is clear that only 4 messages U_1, U_2, V_1, and V_2 are required (as in the IC) to maximize sum rate in the low interference regime (as mentioned in Table I). Further, in [2], it is also proved that in the IC, for channel conditions satisfying

$$|h_1(1 + h_2^2P_2) + h_2(1 + h_1^2P_1)| \leq 1, \tag{4}$$

encoding messages U_1, U_2 alone using Gaussian codebooks and treating interference as noise at each receiver is sum-capacity optimal. In [9], this result is extended to the X channel as well. Having shown that $\gamma_i = 0, i \in \{1, 2\}$, the same result also holds for $\mathcal{R}_{GIN}$.

C. Strong Interference

The conditions for strong interference are $1 \leq h_1^2 \leq P_2 + 1, 1 \leq h_2^2 \leq P_1 + 1$. Define $X_1' = h_1X_1, X_2' = h_2X_2$. Now the equation (3) can be rewritten as

$$\begin{aligned} Y_1 &= \frac{X_1'}{h_1} + X_2' + Z_1 \\ Y_2 &= \frac{X_2'}{h_2} + X_1' + Z_2 \end{aligned} \tag{5}$$

In the strong interference regime, $\frac{1}{h_1} \leq 1, \frac{1}{h_2} \leq 1$.Thus, we now have an equivalent GIN in *low interference* corresponding to the each strong interference GIN. $X_2^{'}$ now carries the *direct* messages to Rx_1 and $X_1^{'}$ carries the *cross private* message to Rx_1. The roles of U_i and W_i, $i \in \{1, 2\}$ interchange from their respective roles in the *low interference* regime. Therefore, $\alpha_i = 0$ (i.e., U_1 and U_2 are not necessary) for maximizing the sum rate.

D. Very Strong Interference

In this case, we can make the following observations:
(1) The conclusions for strong interference that $\alpha_i = 0$ holds here as well.

(2) For the model (5) given above, let $P_1' = \text{var}(X_1') = h_1^2P_1$, $P_2' = \text{var}(X_2') = h_2^2P_2$, $h_1^{'} = 1/h_1$, and $h_2^{'} = 1/h_2$. We already know that, if

$$|h_1^{'}(1 + h_2^{'2}P_2^{'}) + h_2^{'}(1 + h_1^{'2}P_1^{'})| \leq 1 \tag{6}$$

then this corresponds the sub-region in low interference discussed earlier. In this region, only messages W_1 and W_2 are sufficient to maximize the sum rate. The condition can be rewritten in terms of the original channel and power variables are

$$\left|\frac{1 + P_2}{h_1} + \frac{1 + P_1}{h_2}\right| \leq 1. \tag{7}$$

Thus, there is a sub-region within the very strong interference satisfying the above condition where only the 2 messages W_1 and W_2 are necessary to maximize sum rate in $\mathcal{R}_{GIN}$.

VI. SUM-CAPACITY IN THE MIXED INTERFERENCE REGION

In the previous section, we proved that in the mixed interference region the achievable sum rate is maximized by transmitting only 2 messages to one of the receivers, i.e., a MAC at Rx_2 (if $h_1 \geq 1, 0 \leq h_2 \leq 1$) or MAC at Rx_1 (if $h_2 \geq 1, 0 \leq h_1 \leq 1$).

Now, we ask the following question: *Are there channel conditions such that this MAC sum rate is the sum capacity for the GIN?* Some known outerbounds on sum capacity like the MIMO bound (both Tx and Rx cooperation) [19], and the 2×1 MIMO Gaussian BC bound (Tx cooperation) [20], are larger than this MAC sum rate in the mixed interference region. Here, we establish the sum-rate wise optimality of the

MAC sum-rate in some sub-regions of the mixed interference region.

We describe these results for the X channel (for simplicity), i.e., we consider only messages U_1, U_2, W_1, and W_2. The addition of common messages in the GIN case can be shown not to improve the sum rate. We discuss only the $h_1 \geq 1, 0 \leq h_2 \leq 1$ case. Similar results can be shown for the $h_2 \geq 1, 0 \leq h_1 \leq 1$ case as well.

Theorem 4: The sum capacity is achieved for the following three sub-regions of the mixed interference region.

1) $h_1 h_2 = 1$
2) $h_1^2 \geq 1 + P_2, 0 \leq h_2 \leq 1$
3) $h_2^2 \leq \frac{1}{1+h_1^2 P_1}, h_1 \geq 1$

Proof:

1) $h_1 h_2 = 1$: Multiplying the second equation in (3) by h_2 and using $h_1 h_2 = 1$, we get

$$\begin{aligned} h_2 Y_2 &= h_2 X_2 + h_1 h_2 X_1 + h_2 Z_2 \\ &= X_1 + h_2 X_2 + h_2 Z_2. \end{aligned} \tag{8}$$

$h_2 Z_2$ has a variance $h_2^2 \leq 1$ i.e. Rx_2 is a better receiver to even the private messages intended for Rx_1 apart from decoding its own signals. Thus, Y_1 is a degraded version of Y_2 or $(X_1, X_2) \rightarrow Y_2 \rightarrow Y_1$ and sum-rate is maximized by the MAC sum-rate $I(X_1, X_2; Y_2)$.

2) $h_2^2 \geq 1+P_2, 0 \leq h_2 \leq 1$: This proof is in three parts. (i) When Rx_2 has an appropriately chosen side information S_2, we show that, if U_1 is a null message (denoted $U_1 = \phi$), then $W_2 = \phi$ to achieve sum capacity, i.e., the sum capacity is achieved by MAC transmission to Rx_2. (ii) Then, we show that $U_1 = \phi$ by showing that U_1 is decodable at Rx_2 making use of S_2. (iii) Finally, we show that side information S_2 is redundant and not required to maximize the sum-capacity.

(i) Assume that $U_1 = \phi, R_{U_1} = 0$, i.e. $X_1 = f_1(W_1)$.

$$\begin{aligned} nS =& n(R_{U_1} + R_{W_1} + R_{U_2} + R_{W_2}) \\ =& n(R_{W_1} + R_{U_2} + R_{W_2}) \\ =& h(W_1) + n(R_{U_2} + R_{W_2}) \\ =& I(W_1; Y_2^n, S_2^n) + h(W_1|Y_2^n, S_2^n) + n(R_{U_2} + R_{W_2}) \\ \overset{(a)}{\leq}& I(W_1; Y_2^n, S_2^n) + n\epsilon + n(R_{U_2} + R_{W_2}) \\ \overset{(b)}{\leq}& I(X_1^n; Y_2^n, S_2^n) + n\epsilon + n(R_{U_2} + R_{W_2}) \\ \overset{(c)}{\leq}& I(X_1^n; Y_2^n, S_2^n) + \min_{\rho} I(X_2^n; Y_1^n, Y_2^n, S_2^n|X_1^n) + n\epsilon \\ =& I(X_1^n; Y_2^n, S_2^n) + I(X_2^n; Y_2^n, S_2^n|X_1^n) \\ &+ \min_{\rho} I(X_2^n; Y_1^n|X_1^n, Y_2^n, S_2^n) + n\epsilon \\ \overset{(d)}{\leq}& I(X_1^n; Y_2^n, S_2^n) + I(X_2^n; Y_2^n, S_2^n|X_1^n) \\ &+ \min_{\rho} I(X_2^n; Y_1^n|X_1^n, Y_2^n) + n\epsilon \\ \overset{(e)}{=}& I(X_1^n, X_2^n; Y_2^n, S_2^n) + n\epsilon \end{aligned}$$

(a) follows from Fano's inequality, (b) follows from X_1^n being a deterministic function of W_1, (c) is obtained as a valid outerbound for $R_{U_2} + R_{W_2}$ by decoding U_2, W_2 with receiver cooperation with X_1 known. Since the capacity depends only the marginal distributions $p(y_i/x_1, x_2)$, any correlation between Z_1, Z_2 does not affect the capacity i.e. the minimizing over the correlation ρ is a valid outerbound. (d) follows from conditioning reduces entropy, (e) follows from $\rho = h_2$ being the minimizer. Note that choosing $\rho = h_2$ requires $0 \leq h_2 \leq 1$. The minimization with respect to ρ is explained below.

$$\begin{aligned} & I(X_2^n; Y_1^n|Y_2^n, X_1^n) \\ &\quad = h(Y_1^n|Y_2^n, X_1^n) - h(Y_1^n|Y_2^n, X_1^n, X_2^n) \\ &\quad = h(Y_1^n|Y_2^n, X_1^n) - h(Z_1^n|Z_2^n) \\ &\quad = h(h_2 X_2^n + Z_1^n|X_2^n + Z_2^n) - h(Z_1^n|Z_2^n) \\ &\quad \overset{(g)}{\leq} nh(h_2 X_{2G} + Z_1|X_{2G} + Z_2) - nh(Z_1|Z_2) \\ &\quad = nI(X_{2G}; h_2 X_{2G} + Z_1|X_{2G} + Z_2) \\ &\quad = nI(X_{2G}; X_{2G} + Z_1/h_2|X_{2G} + Z_2) \end{aligned} \tag{9}$$

where $X_{iG} \sim N(0, P_i)$. (g) follows from *Lemma 1* in [2] and the assumption we make that $E[z_{1i} z_{2j}] = 0, \forall i \neq j$ and $E[z_{1i} z_{2j}] = \rho, i = j$. Remember that Z_is are already i.i.d in the GIN model. *Lemma 8* in [2] says that when X, N_1, N_2 are Gaussian with X being independent of the two zero-mean random variables N_1, N_2 then $I(X; X + N_1|X + N_2) = 0$ when $E[N_1 N_2] = E[N_2^2]$. Thus, equation (9) reduces to 0 when $\frac{\rho}{h_2} = 1 \Rightarrow \rho = h_2$.

(ii) Whenever U_1 is decodable at Rx_2, $U_1 = \phi$ (without loss of sum-rate). Consider $S_2 = B_{2G} - X_2$ where $B_{2G} \sim \mathcal{N}(0, P_2)$ and independent of X_1. If $I(X_1; Y_2, S_2) \geq I(X_1; Y_1|X_2)$ for all distributions $f(x_1)f(x_2)$ for X_1, X_2, then X_1, including U_1, is completely decodable from $\{Y_2, S_2\}$. Since X_1 is independent of S_2, we have

$$\begin{aligned} I(X_1; & Y_2, S_2) = I(X_1; S_2) + I(X_1; Y_2|S_2) \\ &= I(X_1; Y_2|S_2) = h(Y_2|S_2) - h(Y_2|S_2, X_1) \\ &= h(h_1 X_1 + X_2 + Z_2|B_{2G} - X_2) \\ &- h(X_2 + Z_2|B_{2G} - X_2, X_1) \\ &\overset{(h)}{=} h(h_1 X_1 + B_{2G} + Z_2|B_{2G} - X_2) \\ &- h(B_{2G} + Z_2|B_{2G} - X_2) \\ &= I(X_1; \hat{Y_2}|S_2) = h(X_1|S_2) - h(X_1|\hat{Y_2}, S_2) \\ &= h(X_1) - h(X_1|\hat{Y_2}, S_2) \\ &\geq h(X_1) - h(X_1|\hat{Y_2}) = I(X_1; \hat{Y_2}) \end{aligned}$$

where $\hat{Y_2} = h_1 X_1 + B_{2G} + Z_2$. (h) follows from adding $B_{2G} - X_2$ and independence of X_1 from other terms. Thus, $I(X_1; Y_2, S_2) \geq I(X_1; \hat{Y_2})$.

Since $h_1^2 \geq 1 + P_2$,

$$\mathrm{Var}\left(\frac{B_{2G} + Z_2}{h_1}\right) \leq \mathrm{Var}(Z_1) = 1$$

Therefore, $I(X_1; \hat{Y_2}) \geq I(X_1; Y_1|X_2)$ because (a) we can add independent gaussian noise of appropriate variance to $\hat{Y_2}$ such that total noise variance is 1 and (b) use data processing inequality. Combining the two results,

$$I(X_1;Y_2,S_2) \geq I(X_1;\hat{Y_2}) \geq I(X_1;Y_1|X_2) \tag{10}$$

(iii) Now,

$$\begin{aligned} nS &\leq I(X_1^n,X_2^n;Y_2^n,S_2^n) \\ &= I(X_2^n;Y_2^n,S_2^n) + I(X_1^n;Y_2^n,S_2^n|X_2^n) \\ &\overset{(i)}{=} I(X_2^n;Y_2^n) + I(X_2^n;S_2^n|Y_2^n) + I(X_1^n;Y_2^n|X_2^n) \\ &= I(X_1^n,X_2^n;Y_2^n) + I(X_2^n;S_2^n|Y_2^n) \end{aligned}$$

(i) follows from $I(X_1^n;S_2^n|X_2^n,Y_2^n) = 0$ since X_1 is independent of S_2 given X_2,Y_2.

$$\begin{aligned} &I(X_2^n;S_2^n|Y_2^n) = h(S_2^n|Y_2^n) - h(B_{2G}^n - X_2^n|Y_2^n,X_2^n) \\ &= h(S_2^n + Y_2^n|Y_2^n) - h(B_{2G}^n|Y_2^n,X_2^n) \\ &= h(\hat{Y_2}^n|Y_2^n) - h(B_{2G}^n|X_2^n) \\ &\leq h(h_1X_1^n + B_{2G}^n + Z_2^n|h_1X_1^n + X_2^n + Z_2^n) \\ &\overset{(j)}{\leq} nh(h_1X_{1G} + B_{2G} + Z_2|h_1X_{1G} + X_{2G} + Z_2) \\ &\overset{(k)}{=} 0 \end{aligned}$$

where $X_{iG} \sim N(0,P_i)$. (j) follows from *Lemma 1* in [2] and (k) follows from the freedom to choose B_{2G} since it is only a side information and we choose $\rho_{B_{2G}X_{2G}} = 1$. Thus we have

$$\begin{aligned} nS &\leq I(X_1^n,X_2^n;Y_2^n) \qquad (11) \\ &\leq nI(X_{1G},X_{2G};Y_{2G}) \qquad (12) \end{aligned}$$

where Y_{2G} is Y_2 with X_1,X_2 being Gaussian distributed i.e. $Y_{2G} = X_{2G} + h_1X_{1G} + Z_2$. The above follows from gaussian input optimality for gaussian MAC. The proof extends from X channel to the GIN directly since common messages *alone* (with $U_1,W_2 = \phi$) reaching Rx_1 do not improve the overall sum-rate.

3) $h_2^2 \leq \frac{1}{1+h_1^2P_1}, h_1 \geq 1$: As in the previous case, this proof is also in three parts. Since it follows similar argumants, we point out only the differences here. (i) With side information S_2 at Y_2, we can show that if W_2 is a null message (denoted $W_2 = \phi$), then $U_1 = \phi$ to achieve sum capacity, i.e., the sum capacity is achieved by MAC transmission to Rx_2. (ii) Then, we can show that $W_2 = \phi$ by showing that W_2 is decodable at Rx_2. (iii) Finally, we can show that S_2 does not contribute to improving sum-rate and hence redundant.

   (i): On similar lines as in previous case, we have

$$\begin{aligned} nS &= n(R_{U_1} + R_{W_1} + R_{U_2}) \\ &\leq I(U_2;Y_2^n,S_2^n) + n\epsilon + n(R_{U_1} + R_{W_1}) \\ &\leq I(X_2^n;Y_2^n,S_2^n) + n\epsilon \\ &+ \min_\rho I(X_1^n;Y_1^n,Y_2^n,S_2^n|X_2^n) \\ &\leq I(X_1^n,X_2^n;Y_2^n,S_2^n) \\ &+ \min_\rho I(X_1^n;Y_1^n|Y_2^n,X_2^n) + n\epsilon \\ &\overset{(l)}{=} I(X_1^n,X_2^n;Y_2^n,S_2^n) + n\epsilon \end{aligned}$$

   (l) follows from $\rho = 1/h_1$ being the minimizer, and hence $0 \leq \rho = 1/h_1 \leq 1$. The minimization with respect to ρ is very similar to previous case.

   (ii): We can show $I(X_2;Y_2,S_2) \geq I(X_2;Y_1|X_1)$ for all distributions $f(x_1)f(x_2)$ on X_1, X_2 with $S_2 = h_1B_{1G} - h_1X_1$. Therefore, X_2, including W_2, is completely decodable from Y_2.

   (iii) Then, we can prove that $nS \leq nI(X_{1G},X_{2G};Y_{2G})$ by using $\text{Var}\left(\frac{Z_1}{h_2}\right) = \frac{1}{h_2^2} \geq h_1^2P_1 + 1 = \text{Var}\left(h_1X_1 + Z_2\right)$. Finally, we can show that S_2 does not improve the sum-rate choosing $\rho_{B_{1G}X_1} = 1$.

■

Similar results for the other mixed interference region $h_1 \leq 1, h_2 \geq 1$ can be obtained the same way. In summary, we have 5 channel conditions (in the mixed interference region) under which the sum-capacity is achieved by just 2 of the 6 messages. This partly addresses a question in [9] regarding the possibility of sufficiency of 2 messages for sum-capacity in regions other than some low interference and very high interference regions.

VII. CONCLUSIONS

Using an achievable rate region similar to the Han-Kobayashi region, we obtain an achievable sum rate for a 2 × 2 GIN. We determine that at most 4 (out of 6) messages are sufficient for maximizing the sum rate within this rate region for all channel conditions. Also, in no case is more than one private message transmitted from any transmitter. It is also observed that 2 messages are sufficient in several cases – mixed interference, and sub-regions of low and very strong interference regions. We also show that sum capacity is achieved using only 2 messages for a subset of the mixed interference conditions. In this case, MAC transmission to one of the receivers achieves sum capacity.

APPENDIX A
PROOF OF THEOREM 1

We know $S = R_{U_1} + R_{V_1} + R_{W_1} + R_{U_2} + R_{V_2} + R_{W_2}$ is the sum of 6 different rates. The rate region constraints are constraints on the sum of 1 or 2 or 3 or 4 of these rates. The 15 constraints at each receiver comprise of 4 single rate constraints, 6 on sum of 2 rates, 4 on sum of 3 rates and one on the sum of 4 rates. In order to obtain a bound on S, we can choose 2 or more constraints from the 30 available constraints appropriately.

First, we observe that only one constraint needs to be chosen from each group of 15 constraints (i.e. for each receiver). This is because:

- The messages are independent given Q by assumption.
- If more than one constraint is chosen from the same group (corresponding to the same receiver), a single tighter constraint can be obtained in the following manner. If 2 constraints are chosen from equation (1) corresponding to 2 disjoint subsets C_1 and C_2 of M_1, we get the sum constraint $I(C_1;Y_1|\bar{C}_1,Q) + I(C_2;Y_1|\bar{C}_2,Q)$. However, $I(C_1 \bigcup C_2;Y_1|\overline{C_1 \bigcup C_2},Q)$ is a tighter bound (due to the

independent messages assumption) and is also one of the 15 constraints.

Now, there are only 4 possible combinations of 2 constraints with one from each group of 15 constraints. These are the 4 stated bounds T_1, T_2, T_3, and T_4 in the theorem. A similar approach is also used in [5] for reducing the number of sum constraints in an interference channel setting.

APPENDIX B
PROOF OF THEOREM 2

In order to prove statement (1), it is sufficient to show that any one of the T_i's is less than or equal to $C(h_1^2 P_1 + P_2)$. This is because (i) any bound on a T_i is also a bound on S, and (ii) we know that $C(h_1^2 P_1 + P_2)$ can be achieved using messages U_2 and W_1 alone.

We can show that $T_1 \leq C(h_1^2 P_1 + P_2)$ for any $\alpha_i, \beta_i, \gamma_i$ and $0 \leq h_2 \leq 1$, $h_1 \geq 1$. Proving $T_1 \leq C(h_1^2 P_1 + P_2)$ can be shown (using the monotonicity of the $\log$ function) to be equivalent to showing

$$\frac{1 + \alpha_1 P_1 + \gamma_1 P_1 + h_2^2 \alpha_2 P_2 + h_2^2 \gamma_2 P_2}{(1 + \gamma_1 P_1 + h_2^2 \alpha_2 P_2)(1 + \gamma_2 P_2 + h_1^2 \alpha_1 P_1)} \leq 1.$$

This is the same as showing

$$\alpha_1 P_1 + h_2^2 \gamma_2 P_2 \leq (h_1^2 \alpha_1 P_1 + \gamma_2 P_2)(1 + \gamma_1 P_1 + h_2^2 \alpha_2 P_2).$$

This condition is true for $0 \leq h_2 \leq 1$, $h_1 \geq 1$.

APPENDIX C
PROOF OF THEOREM 3

Comparison of t_1 *and* $t_1^{'}$*:*

$$t_1^{'} = C\left(\frac{\alpha_1 P_1}{1 + h_2^2 \alpha_2 P_2}\right) + C\left(\frac{P_2 + h_1^2 \bar{\alpha}_1 P_1}{1 + h_1^2 \alpha_1 P_1}\right)$$

Proving $t_1 \leq t_1^{'}$ can be shown (using the monotonicity of the $\log$ function) to be equivalent to showing

$$\frac{A_1 + \gamma_1 P_1 + h_2^2 \gamma_2 P_2}{(A_2 + \gamma_1 P_1)(A_3 + \gamma_2 P_2)} \leq \frac{A_1}{A_2 A_3},$$

where $A_1 = 1 + \alpha_1 P_1 + h_2^2 \alpha_2 P_2$, $A_2 = 1 + h_2^2 \alpha_2 P_2$, and $A_3 = 1 + h_1^2 \alpha_1 P_1$. Equivalently, we need to show

$$A_2 A_3 (\gamma_1 P_1 + h_2^2 \gamma_2 P_2) \leq \gamma_1 P_1 A_1 A_3 + \gamma_2 P_2 A_1 A_2 + \gamma_1 P_1 \gamma_2 P_2 A_1.$$

This is shown by comparing the first 2 terms using: (a) $A_1 \geq A_2$, (b) $A_1 \geq A_3$ when $0 \leq h_1 \leq 1$, and (c) $0 \leq h_2 \leq 1$.
Comparison of t_2 *and* $t_2^{'}$*:* This is similar to the comparison of t_1 and $t_1^{'}$ expect that the indices 1 and 2 are interchanged in the expressions for t_2 and $t_2^{'}$ when compared with t_1 and $t_1^{'}$.
Comparison of t_3 *and* $t_3^{'}$*:*

$$t_3^{'} = C\left(\frac{\alpha_1 P_1 + h_2^2 \bar{\alpha}_2 P_2}{1 + h_2^2 \alpha_2 P_2}\right) + C\left(\frac{\alpha_2 P_2 + h_1^2 \bar{\alpha}_1 P_1}{1 + h_1^2 \alpha_1 P_1}\right)$$

Clearly, t_3 is always less than or equal to $t_3^{'}$ since only denominator is reduced (by setting $\gamma_1 = \gamma_2 = 0$) in both the arguments for $C(.)$ in $t_3^{'}$.
Comparison of t_4 *and* $t_4^{'}$*:*

$$t_4^{'} = C\left(\frac{P_1}{1 + h_2^2 \alpha_2 P_2}\right) + C\left(\frac{P_2}{1 + h_1^2 \alpha_1 P_1}\right).$$

Proving $t_4 \leq t_4^{'}$ can be shown (using the monotonicity of the $\log$ function) to be equivalent to showing

$$\left(\frac{A_1 + h_2^2 \gamma_2 P_2}{A_2 + \gamma_2 P_2}\right)\left(\frac{A_3 + h_1^2 \gamma_1 P_1}{A_4 + \gamma_1 P_1}\right) \leq \frac{A_1}{A_2} . \frac{A_3}{A_4},$$

where $A_1 = 1 + P_1 + h_2^2 \alpha_2 P_2$, $A_2 = 1 + h_2^2 \alpha_2 P_2$, $A_3 = 1 + P_2 + h_1^2 \alpha_1 P_1$, and $A_4 = 1 + h_1^2 \alpha_1 P_1$. This is true for $0 \leq h_1, h_2 \leq 1$.

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