LDPC codes for OFDM over an Inter-symbol Interference Channel

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Outline

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Background on LDPC codes

- Low Density Parity Check (LDPC) codes
  - linear codes with sparse parity-check matrices
  - simple definition, capacity-approaching performance
- LDPC analysis and design
  - large ensembles of codes - all with same performance
  - random code from ensemble performs close to average
- How is the ensemble specified?
  - weights of the columns and rows of the parity-check matrix
  - weights are collected into weight distribution polynomials

Analysis and Design Tools for LDPC Codes

Study average performance of ensemble of codes whose parity-check matrices have the same weight distribution
• Message-passing decoders: practical, iterative
  • performance of ensemble is studied under message-passing decoding

• Threshold phenomenon
  • threshold = SNR* ⇒ SNR > SNR* will result in successful decoding
  • block-length → ∞, iterations → ∞

• Density evolution
  • tool to determine threshold

Study of LDPC codes in a new system involves...

developing a density evolution algorithm and determination of threshold
Threshold phenomenon

Performance of irregular LDPC codes; Rate=1/2; AWGN channel

Bit Error Rate vs. $E_b/N_0$ (dB)

- $n=10^3$
- $n=10^4$
- $n=10^5$
- Uncoded

Capacity and Threshold boundaries
OFDM

- The channel model

\[ \hat{c} = H.c + N. \]

- Binary Input alphabet. BPSK modulation.

- Assumptions:
  - A codeword is distributed over a single OFDM symbol
  - The blocklength of the code \((N_c)\) tend to infinity

- In the limit, there is no cyclic prefix overhead
Prior Work on LDPC Codes in an OFDM System

- Prior work on LDPC over OFDM
  - Mannoni et al: mixture PDF analysis and optimization of degree distribution
  - Baynast et al: positioning of information bits in OFDM subcarriers
- Prior work on LDPC over ISI
  - Kavcic et al: LDPC codes over binary-input ISI channels with BCJR
- Previous theoretical works employ a Gaussian mixture density analysis for threshold estimation
- No rigorous proof for the existence of threshold in OFDM systems
Our Work

- Propose a rigorous density evolution
- Existence of LDPC thresholds
- Method for threshold estimation
- Comparison of LDPC thresholds with OFDM capacity
- Comparisons between the time-domain BCJR algorithm proposed by Kavciv et al.
- Mercury/Waterfilling power allocation to improve the OFDM capacity and LDPC thresholds
LDPC Codes: Regular and Irregular

- Regular LDPC Codes
  - H matrix with constant column weight \( w_c \) and constant row weight \( w_r \)
  - Notation: \((n, w_c, w_r)\) regular code

- Irregular LDPC Codes
  - Column weights (row weights) are not equal
  - Bit node degree distribution \( \lambda(x) = \sum_{i=2}^{d_v} \lambda_i x^{i-1} \)
    \( \lambda_i \): the fraction of all edges connected to variable nodes of degree \( i \)
  - Check node degree distribution \( \rho(x) = \sum_{j=2}^{d_c} \rho_j x^{j-1} \)
    \( \rho_j \): the fraction of all edges connected to check nodes of degree \( j \)
  - Notation: \((n, \lambda, \rho)\)
Density Evolution

- Tracks the evolution of the pdf of the messages
- Initial Message: LLR of the received value
  For AWGN channel, initial PDF of the messages $f_0 \equiv \mathcal{N}(\frac{2}{\sigma^2}, \frac{4}{\sigma^2})$
- PDF of the messages after $l$ rounds of message passing is calculated recursively

$$f_l = f_0 \otimes \lambda(\rho(f_{l-1}))$$

$$\lambda(f) := \sum_i \lambda_i f \otimes (i-1), \quad \rho(f) := \sum_i \rho_i f \tilde{\otimes} (i-1)$$

- Average probability of error after $l^{th}$ iteration at given SNR:

$$Pr(error)^l = Pr(message < 0) + \frac{1}{2} Pr(message = 0)$$
Density Evolution: Conditions

- Channel Symmetry
  \[ p(y_t = q|x_t = 1) = p(y_t = -q|x_t = -1). \]

- Decoder Symmetry
  - Variable node symmetry
  - Check node symmetry

- Advantage: Error probability becomes independent of codeword

- Symmetry of Message PDF
  \[ f_i(x) = e^x f_i(-x). \]
Channel Symmetry
- OFDM channel → Parallel AWGN channels
- Each channel is symmetric.
  \[ p_{Z_i|c_i}(z_i|c_i = 1) = p_{Z_i|c_i}(-z_i|c_i = -1). \]

Analysis can be restricted to the All-one Codeword

LLR density in the \( i \)th channel:
  \[ U_i \sim \mathcal{N}\left(\frac{4|H[i]|^2}{\sigma^2}, \frac{8|H[i]|^2}{\sigma^2}\right). \]

LLR distribution is symmetric
Interleaving

- How should the bits be assigned to the subcarriers?

- Equivalent to the design of an interleaver
- Is there an optimum assignment?.
- Are we going to analyze the LDPC performance for a given assignment?
  - Gaussian approximation is necessary in the analysis
Random Interleaving

- Concentration Theorem:
  LDPC performance with different random interleaving are concentrated around the average performance
- It is enough to analyze this average performance
- Eliminates the need for Gaussian approximation in the analysis
Concentration Theorem

- LDPC performance with different random interleaving are concentrated around the average performance.
- Define: $p_{H_i}^l$ = probability of incorrect message along an edge at the $l$th iteration when the interleaver chosen uniformly at random is $H_i$.
- Define: Error concentration probability $\overline{p} = \frac{1}{N!} \sum_{i=1}^{N!} p_{H_i}^l$
- Theorem:

$$P \left( \left| p_{H_i} - \overline{p} \right| \geq \frac{\epsilon}{2} \right) \leq 2e^{-\beta \epsilon^2 n}.$$ 

- It is enough to analyze this average performance.
- Eliminates the need for Gaussian approximation in the analysis.
The algorithm

Consider a degree distribution pair \((\lambda, \rho)\) and transmission over an OFDM channel with \(N_c\) subcarriers with code of blocklength \(n = N_c\), with associated L-densities \(\tilde{f}_i, i \in \{1, 2, \ldots, N_c\}\). Define

\[
f_0 = \frac{1}{N_c} \sum_{i=1}^{N_c} \tilde{f}_i,
\]

then for \(l \geq 1\),

\[
f_l = f_0 \otimes \lambda (\rho (f_{l-1})) ,
\]

• Monotonicity and Threshold
  • The update equations: Same as AWGN
  • Same monotonicity argument
  • Existence of threshold!!
We let the number of subcarriers $N_c$ tend to infinity

LLR distribution depends on the DTFT of the channel impulse response $H(e^{j\omega})$

$$f(u, \omega) = \frac{\sigma}{4|H(e^{j\omega})|\sqrt{\pi}} \exp \left[ -\frac{(\sigma^2 u - 4|H(e^{j\omega})|^2)^2}{16|H(e^{j\omega})|^2\sigma^2} \right]$$

$$H(e^{j\omega}) = \sum_{i=-\infty}^{\infty} h[i]e^{-j\omega i}$$

LLR distribution is now a continuous function of the angular frequency $\omega$

Summation changes to an integral

$$f_0(u) = \frac{1}{2\pi} \int_{0}^{2\pi} f(u, \omega).d\omega$$
The function \( f(u, \omega) \) is not always well behaved

- Problems in channels with spectral nulls
- New approach to calculate the \( f_0(u) \)
- Using the idea of characteristic function

\[
\begin{align*}
  f(u, \omega) & \rightarrow f_0(u) \\
  \hat{f}(t, \omega) & \rightarrow \hat{f}(t)
\end{align*}
\]
Threshold Estimation

- Characteristic function:

\[ \hat{f}(t, \omega) := \int_{-\infty}^{\infty} f(u, \omega) e^{jut} du \]

\[ = \exp \left[ -\frac{4|H(e^{j\omega})|^2 t^2}{\sigma^2} + j\frac{4|H(e^{j\omega})|^2 t}{\sigma^2} \right] \]

- Advantage: A well behaved characteristic function obtained analytically

\[ \hat{f}(t) := \frac{1}{2\pi} \int_{0}^{2\pi} \hat{f}(t, \omega) d\omega \]

\[ f_0(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t) e^{-jut} dt \]
Results

- Thresholds for different rate regular and irregular LDPC codes
- Validation by simulation
- Comparison with OFDM capacity
- Comparison with LDPC threshold over a binary ISI channel with BCJR equalization
Thresholds: Channel without spectral null

Channel: \( h_2[i] = [0.800, 0.600] \)
Thresholds: Channel with spectral null

- Channel: \( \{ h_1[i] \} = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \).

![Channel with spectral Null](image)
Mercury/Waterfilling Power allocation

- The optimum power allocation for parallel Gaussian channels with arbitrary input constellation
- Channel: \( \{ h_1[i] \} = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \).

Applying Mercury/waterfilling for better LDPC thresholds
LDPC thresholds with Mercury/Waterfilling Power allocation

- Channel: \( \{ h_1[i] \} = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \).

![Graph showing LDPC thresholds with Mercury/Waterfilling Power allocation]
Conclusions

- Developed a rigorous density evolution for binary-input OFDM and proved the existence of thresholds
  - LDPC thresholds are very close to OFDM capacity at higher rates
- Compared OFDM-BPSK capacity and ISI-BPSK capacity
  - At higher rates, ISI-LDPC thresholds are much better than OFDM-LDPC thresholds
- Mercury/Waterfilling power allocation over OFDM subcarriers
  - Again, LDPC thresholds are very close to OFDM capacity
Future work

- Achieving capacity at very low rates
- Optimum bit-loading with Mercury/Waterfilling power allocation to improve capacity and thresholds
- Optimization of irregular LDPC code for OFDM
- Extension to wireless channels
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