Lecture 37: (4 Nov 2008)

So far, we discussed capacity of point-to-point wireless links.

How can multi-terminal systems be studied?

Network information theory / multi-terminal info. theory provides some answers.

In a cellular system, all users communicate only with base-stations.

Therefore, we primarily have 2 kinds of <u>multiuser channels</u>.

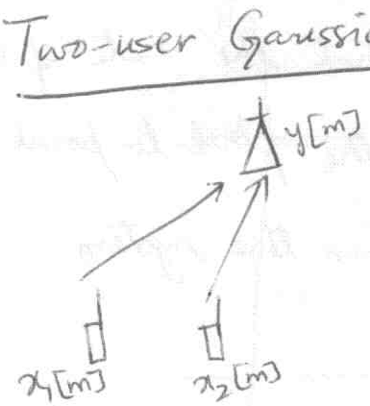
(1) Multiple access channel (uplink) (MAC)

(2) Broadcast channel (downlink) (BC)

First we will consider multiuser channels without fading.

Later, we will consider flat fading.

(I) <u>Two-user Gaussian MAC</u>: (Uplink) AWGN



$$y[m] = \underbrace{x_1[m]}_{\text{Power constraint } P_1} + \underbrace{x_2[m]}_{P_2} + \underbrace{w[m]}_{\mathcal{CN}(0, N_0) \text{ i.i.d}}$$

Let R_1 be the rate for user 1 & R_2 be the rate for user 2.

Now, we have a <u>capacity region</u>.

This is the set of all (R_1, R_2) such that users 1 & 2 can <u>simultaneously</u> communicate <u>reliably</u> at rates R_1 & R_2 respectively.

This capacity region is known for the Gaussian MAC.
(Info-theory result : Achievability proof + converse).

Let us just understand the region, not derive it.

The capacity region consists of all (R_1, R_2) such that

$$R_1 \leq \log\left(1+\frac{P_1}{N_0}\right)$$

$$R_2 \leq \log\left(1+\frac{P_2}{N_0}\right)$$

$$R_1+R_2 \leq \log\left(1+\frac{P_1+P_2}{N_0}\right).$$

(See Appendix B.9 for justification).

Remarks :

① The first 2 conditions above say that the rate of each user cannot exceed the capacity of the point-to-point link with the other user being absent from the system.
(Single-user bounds)

② The third condition says that the sum rate of both users together cannot exceed the capacity of a point-to-point AWGN channel with power constraint P_1+P_2.
(For proof, see info theory books)

→ This is reasonable. If 2 users send independent signals, the aggregate received power will be the sum of the individual powers.

Note that even when user 1 is transmitting at its single-user bound, user 2 can transmit at a non-zero rate.

If $R_1 = \log\left(1+\frac{P_1}{N_0}\right)$

$R_1 + R_2 = \log\left(1+\frac{P_1+P_2}{N_0}\right)$,

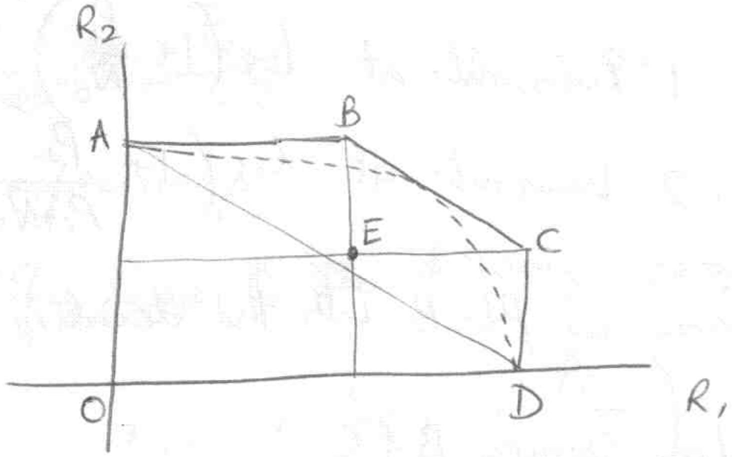
then $R_2 = \log\left(1+\frac{P_1+P_2}{N_0}\right) - \log\left(1+\frac{P_1}{N_0}\right)$

$$= \log\left(\frac{N_0+P_1+P_2}{N_0+P_1}\right)$$

$$= \log\left(1+\frac{P_2}{N_0+P_1}\right).$$

It is as if user 1 is Gaussian interference to user 2.

(3) The capacity region is described by a pentagon.



How are different pts. in this region achieved?

Pt. D   User 2 does not transmit.
User 1 transmits at rate $\log\left(1+\frac{P_1}{N_0}\right)$.

Pt. A   User 1 does not transmit.
User 2 transmits at rate $\log\left(1+\frac{P_2}{N_0}\right)$.

Pts. on line joining A & D

Time-share between the strategies for pts. A & D.

Pt. B

User 2 transmits at $\log\left(1+\frac{P_2}{N_0}\right)$

User 1 transmits at $\log\left(1+\frac{P_1}{P_2+N_0}\right)$

Called a Successive Interference Cancellation (SIC) decoder

- The receiver decodes user 1 assuming user 2 is interference (Gaussian).
- The receiver subtracts user 1's component from the received signal
- The receiver decodes user 2 from the signal obtained after cancelling user 1's signal.

Pt. C

User 1 transmits at $\log\left(1+\frac{P_1}{N_0}\right)$

User 2 transmits at $\log\left(1+\frac{P_2}{P_1+N_0}\right)$

Rx.: Same as Pt. B with the decoding order reversed.

Pts. on the line joining B & C

Time-sharing between the strategies for pts. B & C.

Pt. E (Strictly inside capacity region)

User 1 achieves rate $\log\left(1+\frac{P_1}{P_2+N_0}\right)$

User 2 achieves rate $\log\left(1+\frac{P_2}{P_1+N_0}\right)$

(conventional CDMA with single-user receiver).

4) For pts. in the triangle OAD, only one user transmits at a time. Simultaneous transmission is not necessary.

For B & C, simultaneous transmission is necessary.

Pt. E using conventional CDMA also uses simultaneous transmission.

$\boxed{\text{Lecture 38}}$ (5 Nov 2008).

5) More (R_1, R_2) pts. are achievable without simultaneous transmission than the triangle OAD.

Consider orthogonal multiple access as follows.

Say user 1 uses the channel for α fraction of time

& user 2 — " ———— $1-\alpha$ — " ————

However, while transmitting users 1 & 2 use power $\frac{P_1}{\alpha}$ & $\frac{P_2}{1-\alpha}$. On average, they still use P_1 & P_2 respectively.

This leads to the following possible (R_1, R_2) pairs

$$R_1 = \alpha \log\left(1 + \frac{P_1}{\alpha N_0}\right)$$

$$R_2 = (1-\alpha)\log\left(1 + \frac{P_2}{(1-\alpha) N_0}\right).$$

As α varies from 0 to 1, we get the curve (shown in dotted line) in the figure in the previous page.

This scheme achieves the maximum sum rate possible at one point. $\left(\alpha = \frac{P_1}{P_1+P_2}\right)$

General K-user uplink capacity:

K-user capacity region is described by $2^K - 1$ constraints, one for each non-empty subset S of users.

$$\sum_{k \in S} R_k \leq \log\left(1 + \frac{\sum_{k \in S} P_k}{N_0}\right) \quad \text{for all } S \subset \{1,2,\ldots,K\}.$$

Sum capacity $C_{sum} = \log\left(1 + \frac{\sum_k P_k}{N_0}\right)$.

If P_k's are equal to P, $C_{sum} = \log\left(1 + \frac{KP}{N_0}\right)$.

Note: * As K increases, C_{sum} increases & is unbounded.

* For a CDMA system where other users are treated as noise (no SIC)

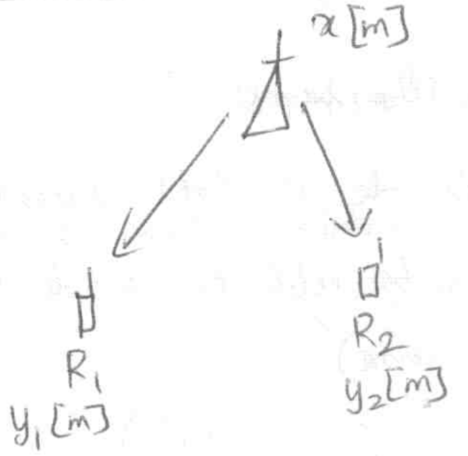
$$\text{sum rate} = K \log\left(1 + \frac{P}{(K-1)P + N_0}\right)$$

$$(\text{For large } K) \approx K \frac{P}{(K-1)P + N_0} \log_2 e \approx \log_2 e.$$

Here, the rate is "interference-limited."

* In a cellular system, we have out-of-cell interferers. If they are not jointly decoded, the capacity is always interference-limited.

II Two-user Gaussian Broadcast Channel (Downlink AWGN)



h_1, h_2 fixed known to transmitter & receivers.

$$y_1[m] = h_1 x[m] + w_1[m]$$

$$y_2[m] = h_2 \underbrace{x[m]}_{\text{Power constraint } P} + \underbrace{w_2[m]}_{\mathcal{CN}(0, N_0) \text{ i.i.d.}}$$

Simple single-user bounds:

$$R_k < \log\left(1 + \frac{P|h_k|^2}{N_0}\right) \quad k = 1, 2.$$

With $R_1 = 0$, $R_2 = \log\left(1 + \frac{P|h_2|^2}{N_0}\right)$ can be achieved.

With $R_2 = 0$, $R_1 = \log\left(1 + \frac{P|h_1|^2}{N_0}\right)$ can be achieved.

By time-sharing, points joining the above 2 points can be achieved.

Can rate-pair (R_1, R_2) outside this triangle be achieved?

Case a: Symmetric case $|h_1| = |h_2|$

case b: General case $|h_1| \neq |h_2|$

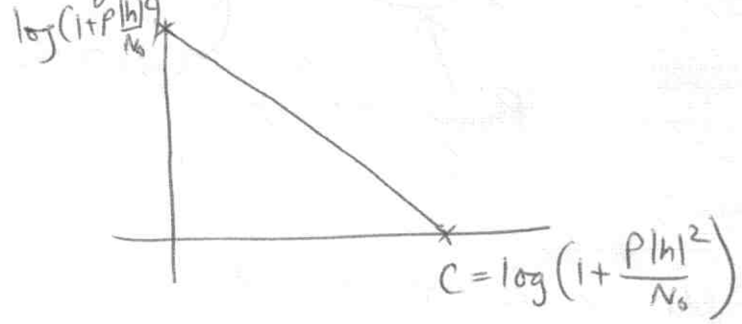
Symmetric case $|h_1| = |h_2| = h$

* SNR for both users is the same

$\Rightarrow$ If user 1 can decode its data succesfully, user 2 should also be able to decode the data of user 1 (and viceversa).

$$\Rightarrow \quad R_1 + R_2 < \log\left(1 + \frac{P|h_1|^2}{N_0}\right)$$

$\Rightarrow$ Triangle below is the capacity region.



Time-sharing achieves all pts. in the region.

* Now, consider superposition $\rightarrow$ (of the 2 user's signals (codewords))

$$x[m] = x_1[m] + x_2[m].$$

If user 1 can decode his data, user 2 can also decode this. Can user 2 improve his rate by subtracting user 1's data?

$$R_1 = \log\left(1 + \frac{P_1|h|^2}{P_2|h|^2 + N_0}\right)$$

where P_1: power allocated for user 1

P_2: power allocated for user 2.

$$= \log\left(1 + \frac{(P_1+P_2)|h|^2}{N_0}\right) - \log\left(1 + \frac{P_2|h|^2}{N_0}\right).$$

$$R_2 = \log\left(1 + \frac{P_2|h|^2}{N_0}\right)$$

$$P_1 + P_2 = P.$$

$$R_1 + R_2 = \log\left(1 + \frac{(P_1+P_2)|h|^2}{N_0}\right)$$

$$= \log\left(1 + \frac{P|h|^2}{N_0}\right)$$

Same region obtained as in the time-sharing case discussed earlier (Triangle).

General case $|h_1| \neq |h_2|$

Consider superposition again & let $|h_1| < |h_2|$.

Since user 2 has the better channel, if user 1 can decode any data, user 2 can also decode it. So, user 2 can perform SIC.

Let P_1 be allocated to user 1
& P_2 —"— user 2

$P = P_1 + P_2$

$$R_1 = \log\left(1 + \frac{P_1|h_1|^2}{P_2|h_1|^2 + N_0}\right)$$

$$\& \quad R_2 = \log\left(1 + \frac{P_2|h_2|^2}{N_0}\right)$$ can be achieved for all P_1, P_2 s.t. $P_1 + P_2 = P$.

$$\left\{ (R_1, R_2) : \begin{array}{l} R_1 < \log\left(1 + \dfrac{\alpha P|h_1|^2}{(1-\alpha)P|h_1|^2 + N_0}\right) \\ R_2 < \log\left(1 + \dfrac{(1-\alpha)P|h_2|^2}{N_0}\right) \end{array} \right\}$$

where $0 \leq \alpha \leq 1$.

It can be shown at superposition coding + SIC achieves all pts. in the capacity region. [Simple converse in Bergmans 1974]

What about orthogonal schemes?

Split power P into P_1 & P_2

Split Deg. of freedom into α & $1-\alpha$ $\quad \alpha \in [0,1]$.

(eg. α represents the fraction of BW to user 1).

$$R_1 = \alpha \log\left(1 + \frac{P_1 |h_1|^2}{\alpha N_0}\right)$$

$$R_2 = (1-\alpha)\log\left(1 + \frac{P_2 |h_2|^2}{(1-\alpha) N_0}\right)$$

can be achieved.

(α, P_1, P_2 can be varied to achieve different pts.)

Superposition coding $>$ Orthogonal scheme

→ Especially when the two users are at very different SNR, superposition can provide a reasonable rate to the strong user, while achieving close to the single-user bound for the weak user.

$\boxed{\text{Lecture 39:}}$ (10 Nov 2008)

General K-user BC

Let $|h_1| \le |h_2| \le \cdots \le |h_K|$

$$R_k = \log\left(1 + \frac{P_k |h_k|^2}{N_0 + \left(\sum_{j=k+1}^{K} P_j\right)|h_k|^2}\right) \qquad k = 1, \ldots, K$$

characterizes the points on the boundary of the capacity region. $P = \sum_{k=1}^{K} P_k$.

$\boxed{\text{Note:}}$

* Sum capacity is maximized by transmitting to the user with highest SNR (not like MAC)

~~Why~~?

Strong user can decode any information sent to the weak user $\Rightarrow$ $R_1 + R_2 <$ cap of strong user

$\Rightarrow$ Sum rate is maximized by transmitting to the strong user.

$$R_1 + R_2 = \alpha \log\left(1 + \frac{\alpha P |h_1|^2}{N_0}\right) + (1-\alpha)\log\left(1 + \frac{(1-\alpha) P |h_2|^2}{\alpha P |h_2|^2 + N_0}\right)$$

(assuming $|h_1| > |h_2|$)

This is maximized for $\alpha = 1$.

Uplink fading channel: Fading MAC.

$$y[m] = \sum_{k=1}^{K} h_k[m]\, x_k[m] + w[m].$$

Let us consider the symmetric fading MAC channel.

$h_k[m]$ i.i.d. $E\left[|h_k[m]|^2\right] = 1$.

Power constraint P for each user

We will study 2 cases as usual (as for point-to-point fading channel)

- Slow fading
- Fast fading

Slow fading MAC:

Let each user transmit at rate R over the symmetric MAC above.

$h_k[m] = h_k$ for all m (slow fading)

For a given channel realization, the capacity region is described by the $2^K - 1$ constraints

$$R|S| < \log\left(1 + \sum_{k \in S} |h_k|^2 \, SNR\right) \text{ for each } S \subseteq \{1, 2, \cdots, K\}$$

Outage occurs when any one or more of these constraints are not satisfied.

Outage Prob.

$$\left\{ P_r(\text{outage}) = P_r\left[\log\left(1 + SNR \sum_{k \in S} |h_k|^2\right) < R|S| \quad \text{for } \underline{\text{some}} \ S \subseteq \{1, 2, \cdots, K\} \right] \right.$$

Outage capacity (symmetric)

For fixed prob. of outage, we can define the ϵ-outage symmetric capacity.

C_ϵ^{sym}: largest rate R such that $P_r(\text{outage}) \leq \epsilon$.

For the asymmetric case, we can define an <u>outage capacity region</u>.

<u>Fast Fading MAC</u>:

For the symmetric fast fading MAC

Sum Capacity

$$C_{sum} = E\left[\log\left(1 + \frac{\sum_{k=1}^{K} |h_k|^2 P}{N_0}\right)\right]$$

As in the point-to-point case, the above ergodic capacity can be achieved with arbitrarily small prob. of error.

$\boxed{\text{Lecture 40:}}$ (11 Nov 2008)

Discussion on capacities of slow & fast fading MAC scenarios:

(1) Slow fading: (Comparison with point-to-point fading channel & orthogonal multiple access.)

First, consider orthogonal multiple access. In this case, the outage event is when

$$\frac{1}{K}\log\left(1+K\,SNR\,|h_k|^2\right) < R \quad \text{for some } k\in\{1,2,\dots,K\}$$

(Each user gets $\frac{1}{K}$ degrees of freedom & can transmit KP over this)

Inter-user interference is eliminated.

$$P_r(\text{outage}) = P_r\left[\frac{1}{K}\log\left(1+K\,SNR\,|h_k|^2\right) < R \text{ for some } k\in\{1,2,\dots,K\}\right]$$

$$= 1-\prod_{k=1}^{K}\underbrace{P_r\left[\frac{1}{K}\log\left(1+K\,SNR\,|h_k|^2\right) > R\right]}_{P_r[\text{No outage for user } k]}$$

If $P_r(\text{no outage for user } k) = \epsilon$,

$$P_r(\text{outage}) = 1-(1-\epsilon)^K$$

$$(\text{For small } \epsilon) \approx K\epsilon.$$

Therefore, largest symmetric ϵ-outage cap

$$C_\epsilon^{sym} \approx \frac{C_{\epsilon/K}(K\cdot SNR)}{K} \quad \text{— (I)}$$

where $C_\epsilon(SNR)$ is the outage capacity for a point to point fading channel at $SNR = SNR$. $C_\epsilon(SNR) = \log(1 + F^{-1}(1-\epsilon) SNR)$.

Can SIC achieve C_ϵ^{sym} larger than (I), i.e., be better than orthogonal multiple access?

<u>Low SNR</u>: For small x, $\log(1+x) \approx x$.

$$\Pr(\text{outage}) = \Pr\left[\log\left(1 + \frac{P \sum_{k \in S} |h_k|^2}{N_0}\right) < R|S| \text{ for some } S \subseteq \{1, 2, \cdots, K\} \right]$$

$2^K - 1$ constraints collapse to these K constraints

$$\approx \Pr\left[|h_k|^2 \frac{P}{N_0} < \overset{\text{nats/s/Hz}}{R} \text{ for some } k \in \{1, 2, \cdots, K\} \right]$$

[This is similar to orthogonal multiple access as shown below]

$$C_\epsilon^{sym} \approx C_{\epsilon/K}(SNR)$$

$$\approx F^{-1}\left(1 - \frac{\epsilon}{K}\right) C_{awgn}$$

$$\approx F^{-1}\left(1 - \frac{\epsilon}{K}\right) SNR$$

$$= \frac{1}{K}\left[F^{-1}\left(1 - \frac{\epsilon}{K}\right) K \cdot SNR \right]$$

$$\approx \frac{1}{K} C_{\epsilon/K}(K \cdot SNR).$$

Similar to orthogonal multiple access.

<u>High SNR</u>: For large x, $\log(1+x) \approx \log x$.

$$\Pr[\text{outage}] \approx \Pr\left[\log \frac{P \sum_{k=1}^{K} |h_k|^2}{N_0} < KR \right]$$

$$= Pr\left[\sum_{k=1}^{K}|h_k|^2 < \frac{e^{KR}}{P/N_0}\right]$$

$$Pr\left[\sum_{k=1}^{K}|h_k|^2 < x\right] \approx \frac{x^K}{K!} \text{ for small } x.$$

$$\frac{x^K}{K!} = \epsilon \Rightarrow x = \epsilon^{1/K}(K!)^{1/K}$$

$$C_\epsilon^{sym} \approx \frac{1}{K}\log\left[\epsilon^{1/K}(K!)^{1/K}\frac{P}{N_0}\right].$$

$$C_\epsilon = \log(1+F^{-1}(1-\epsilon)\,SNR)$$

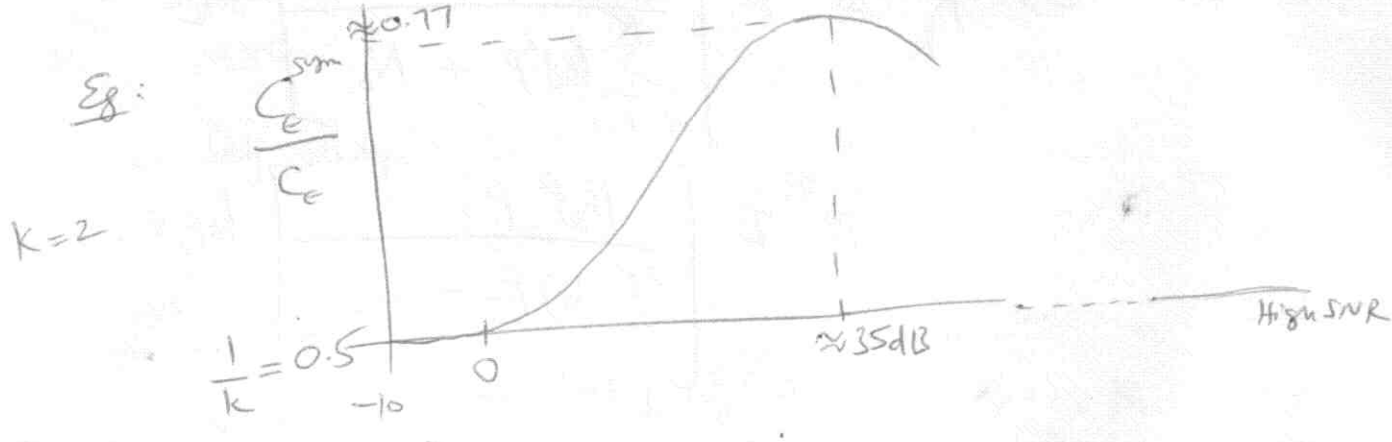
$$\approx \log F^{-1}(1-\epsilon)\,SNR.$$

For large SNR, $\dfrac{C_\epsilon^{sym}}{C_\epsilon} \approx \dfrac{1}{K}.$

(Similar to low SNR case).

For moderate SNRs:

$$\frac{C_\epsilon^{sym}}{C_\epsilon} > \frac{1}{K}$$



$\boxed{\text{Lecture 41}}$ (12 Nov 2008)

(2) Fast fading:

[Symmetric MAC]

$$C_{sum} = E\left[\log\left(1+\frac{\sum_{k=1}^{K}|h_k|^2 P}{N_0}\right)\right].$$

$$E\left[\log\left(1+\frac{\sum_{k=1}^{K}|h_k|^2 P}{N_0}\right)\right] \le \log\left(1+\frac{E\left[\sum_{k=1}^{K}|h_k|^2\right]P}{N_0}\right)$$

Sum capacity of fading MAC.

$$= \log\left(1+\frac{KP}{N_0}\right).$$

Sum capacity of Gaussian MAC.

For large K (number of users), effect of fading goes away.

Consider the rate of k^{th} user (when SIC is used) in the order $1, 2, \ldots, K$.

$$R_k = E\left[\log\left(1+\frac{|h_k|^2 P}{\underbrace{\sum_{i=k+1}^{K}|h_i|^2 P + N_0}}\right)\right]$$

For large K, SINR is low.

$$\Rightarrow R_k \approx E\left[\frac{|h_k|^2 P}{\sum_{i=k+1}^{K}|h_i|^2 P + N_0}\right]\log_2 e$$

$$\approx E\left[\frac{|h_k|^2 P}{(K-k)P + N_0}\right]\log_2 e$$

$$= \frac{P}{(K-k)P+N_0}\log_2 e$$

Rate in an unfaded Gaussian MAC.

$\Rightarrow$ As we add more users, sum capacity is close to sum capacity of Gaussian MAC

For the orthogonal multiple access scheme

$$C_{sum} = \sum_{k=1}^{K} \frac{1}{K} E\left[\log\left(1+\frac{K|h_k|^2 P}{N_0}\right)\right]$$

$$= E\left[\log\left(1+\frac{K|h_k|^2 P}{N_0}\right)\right]$$

$$< E\left[\log\left(1+\frac{\sum_{k=1}^{K}|h_k|^2 P}{N_0}\right)\right] = C_{sum} \text{ with SIC.}$$

As K increases, this C_{sum} has a penalty compared to the Gaussian MAC.

Channel State Information at the transmitter : (Uplink Fading MAC)

- Optimal power allocation can be done to achieve multiuser diversity & get higher capacity than a Gaussian MAC.
- Self-study.

Fading BC :

(Symmetric case)

$$y_k[m] = \underbrace{h_k[m]}_{\text{i.i.d}} \underbrace{x[m]}_{P} + \underbrace{w_k[m]}_{CN(0,N_0)\text{ i.i.d.}} \qquad k=1,2,\ldots,K.$$

Fast fading :

$$R_k < E\left[\log\left(1+\frac{P|h_k|^2}{N_0}\right)\right] \qquad k=1,2,\ldots,K.$$

- Achievable using time-sharing.

→ Extension to asymmetric fading BC difficult.

– Cannot order channels as strong, weak, etc. for superposition coding. (Not degraded)

Channel known at the transmitter:

– Self study.

Multiuser Diversity:

– Self study.