~~Lecture 34~~ (29 Oct 2008)

Capacity of Fading channels: $\rightarrow$ { Flat fading first / Freq-selective fading later

Flat fading

$$y[m] = \underbrace{h[m]}_{\text{Fading process}} x[m] + \underbrace{w[m]}_{CN(0,N_0)\ \text{i.i.d.}}$$

Assume $E\left[|h[m]|^2\right] = 1$ (normalized).

$$E\left[|x[m]|^2\right] \le P$$

Average received SNR $\triangleq \frac{P}{N_0}$.

Assume receiver knows $h[m]$ (channel realization).

Assume transmitter does not know the channel realization.

Assume Transmitter knows the statistical characterization of $h[m]$ (e.g. Rayleigh fading model)

Let us consider 2 cases $\rightarrow$ Slow fading / Fast fading.

"Slow" & "Fast" are defined based on the relationship between coherence time & coding interval.

If coding interval $\ll$ coh. time, we have "slow" fading.

If coding interval $\gg$ coh. time, we have "fast" fading.

In practice, coding interval will be determined by delay requirements.

Slow fading channel:

The channel gain $h[m] = h$ for the coding interval, but h is random.

$$y[m] = h\, x[m] + w[m]$$
$$\downarrow$$
$$CN(0,1)$$

Can we transmit reliably at rate R?

For a given channel realization h, rates above $\log(1+|h|^2 SNR)$ are not possible. Therefore, whatever coding strategy is used by the transmitter, the decoding error probability cannot be made arbitrarily small, i.e., cannot be made smaller than $Pr(\log(1+|h|^2 SNR) < R)$.

This event "$\log(1+|h|^2 SNR) < R$" is referred to as an outage event & the $Pr(\log(1+|h|^2 SNR) < R)$ is the outage probability

$$P_{out}(R) \triangleq Pr(\log(1+|h|^2 SNR) < R)$$

Best strategy:

Encode at the transmitter assuming that the channel is strong enough to support R. If the channel is strong, prob. of error is arbitrarily small. If the channel is weak, we have outage $\Rightarrow$ Overall error prob $= P_{outage}$ (minimum possible)

For arbitrarily low pr. error, capacity $= 0$ $\left\{ \begin{array}{l} \text{whenever} \\ \Pr(\log(1+|h|^2 SNR) < R) > 0 \\ \text{for all } R > 0 \end{array} \right)$

Therefore, we define <u>outage capacity</u>. i.e. $\Pr(\text{deep fade}) > 0$.

ϵ-outage capacity : The maximum rate of transmission R such that $P_{out}(R) \leq \epsilon$.

For Rayleigh fading

exponenial r.v.

$$P_{out}(R) = \Pr\left(|h|^2 < \frac{2^R - 1}{SNR}\right)$$

$$= 1 - e^{-\frac{2^R-1}{SNR}}$$

$$\begin{pmatrix}\text{high} \\ SNR\end{pmatrix} \approx 1 - \left(1 - \frac{2^R - 1}{SNR}\right) = \frac{2^R - 1}{SNR}$$

$\left(e^{-x} \approx 1 - x \text{ for small } x\right)$

$$\propto \frac{1}{SNR}.$$

Error probability $\propto \frac{1}{SNR}$ for large SNR

(Div. gain $= 1$).

Let $F(x) \triangleq \Pr(|h|^2 > x)$.

C_ϵ satisfies $P_{out}(C_\epsilon) = \epsilon$

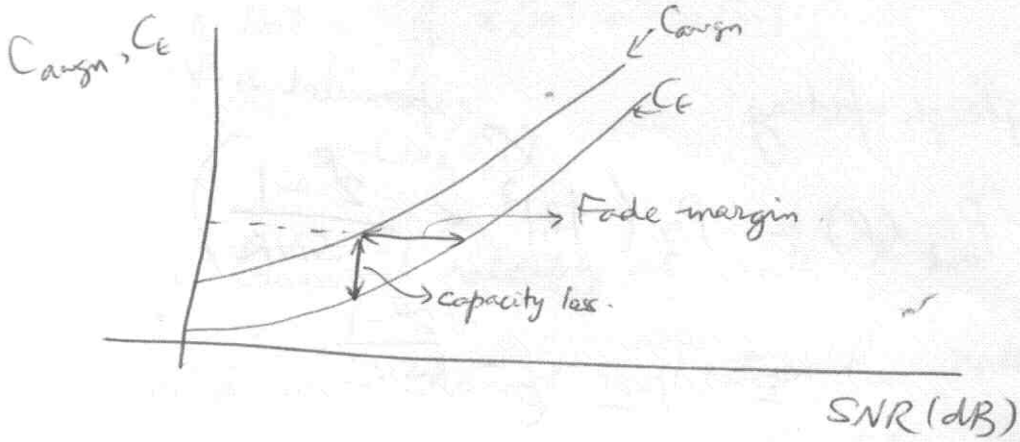
$$P_{out}(C_\epsilon) = 1 - F\left(\frac{2^{C_\epsilon} - 1}{SNR}\right) = \epsilon.$$

$$\Rightarrow 1 - \epsilon = F\left(\frac{2^{C_\epsilon} - 1}{SNR}\right)$$

$$\Rightarrow C_\epsilon = \log\left(1 + F^{-1}(1-\epsilon)\,SNR\right) \quad \text{bits/s/Hz}.$$

ϵ-outage capacity of
A slow fading channel with av. sig. to noise ratio SNR (C_ϵ)

VS
capacity of
An AWGN channel with sig. to noise ratio SNR (C_{awgn})



$C_\epsilon < C_{awgn}$

* For the same rate $C_\epsilon = C_{awgn}$, $10\log\left(\frac{1}{F^{-1}(1-\epsilon)}\right)$ dB more power is needed in the slow fading channel. → called "Fade margin".

* For the same SNR, $C_\epsilon < C_{awgn}$

At high SNR :

$$C_\epsilon \approx \log\left(F^{-1}(1-\epsilon)\,SNR\right)$$
$$\approx \log SNR + \log\left(F^{-1}(1-\epsilon)\right)$$
$$= \log SNR - \log\left(\frac{1}{F^{-1}(1-\epsilon)}\right)$$
$$\approx C_{awgn} - \log\left(\frac{1}{F^{-1}(1-\epsilon)}\right).$$

$(-C_\epsilon + C_{awgn})$ is constant at large SNR.

$$\frac{C_{awgn} - C_\epsilon}{C_{awgn}} \longrightarrow 0 \quad \text{as } SNR \to \infty$$
(because $C_{awgn} \to \infty$)

At low SNR:

$$C_\epsilon \approx F^{-1}(1-\epsilon)\,\mathrm{SNR}$$
$$\approx F^{-1}(1-\epsilon)\,C_{awgn}$$

For small ϵ, $F^{-1}(1-\epsilon) \approx \epsilon \Rightarrow C_\epsilon \approx \epsilon\, C_{awgn}$

$$\frac{C_{awgn} - C_\epsilon}{C_{awgn}} \approx 1-\epsilon \approx 1 \text{ for small } \epsilon.$$

$\rightarrow$ % loss in capacity very high at low SNR & small at high SNR.

$\Rightarrow$ Fading has a more significant impact at low SNR.

$\boxed{\text{Lecture 35:}}$ (31 Oct 2008)

<u>What if we have receive diversity?</u>

$$P_{out}(R) = P\left[\log(1+\|\underline{h}\|^2\,\mathrm{SNR}) < R\right]$$
$$= P\left[\|\underline{h}\|^2 < \frac{2^R-1}{\mathrm{SNR}}\right].$$

$$C_\epsilon = \log(1+F^{-1}(1-\epsilon)\,\mathrm{SNR})$$

where $F(x)$ is now $P\left[\|\underline{h}\|^2 > x\right]$.

At high SNR $\quad P_{out}(R) \approx \dfrac{(2^R-1)^L}{L!\ \mathrm{SNR}^L}$. (Div. gain $= L$)

At high SNR $\quad C_\epsilon \approx \log F^{-1}(1-\epsilon) + \log \mathrm{SNR}$

At low SNR $\quad C_\epsilon \approx F^{-1}(1-\epsilon)\,\mathrm{SNR}$
$$\approx (L!)^{\frac{1}{L}}(\epsilon)^{\frac{1}{L}}\,\mathrm{SNR}$$

For $\epsilon = 0.01$ & $L=2$, $(L!)^{\frac{1}{L}}(\epsilon)^{\frac{1}{L}} = 0.141$

For $L=1$, $\epsilon = 0.01$, at low SNR

$$C_\epsilon \approx 1\% \text{ of } C_{awgn}.$$

With extra rx-antenna, significant improvement at low SNR,

$$C_\epsilon \approx 14\% \text{ of } C_{awgn}$$

Receive antennas $\rightarrow$ power gain
& diversity gain

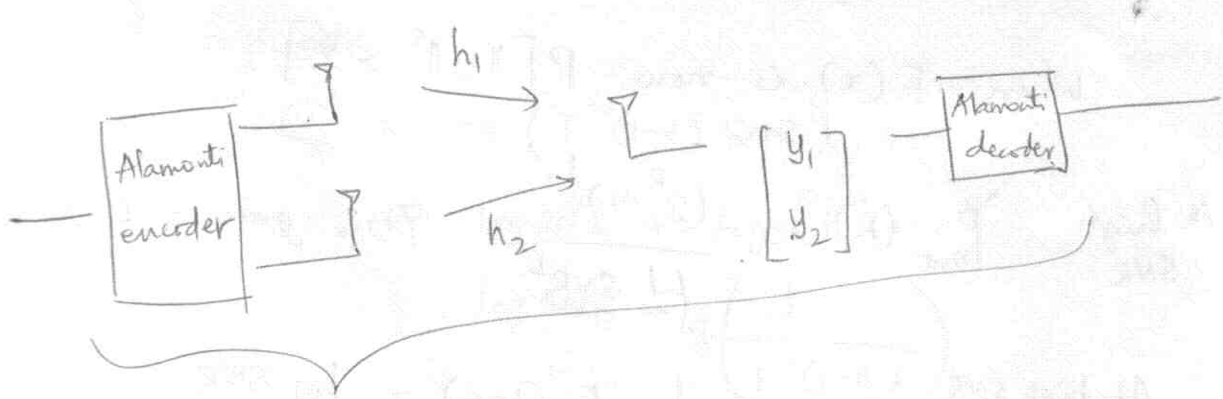
Transmit diversity

If we have multiple antennas at the transmitter, and the channel is known at the transmitter, then for a given $\underline{h}$ the capacity is $\log(1 + \|\underline{h}\|^2 \text{SNR})$, the same as that for a system with multiple receiver antennas.

$\Rightarrow$ results are same as for receive diversity.

If the transmitter does not know the channel, what happens?

Let us look at a 2×1 system & Alamouti scheme first.



Equivalent 1×1

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

$$\begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2^* \end{bmatrix}$$

$$r_1 = h_1^* y_1 + h_2 y_2^* = (|h_1|^2 + |h_2|^2)\, x_1 + (h_1^* w_1 + h_2 w_2^*)$$

$$r_2 = h_2^* y_1 - h_1 y_2^* = (|h_1|^2 + |h_2|^2)\, x_2 + (h_2^* w_1 - h_1 w_2^*)$$

Equivalent 1×1 model

$$r = (|h_1|^2 + |h_2|^2)\, x + n \qquad n \sim \mathcal{CN}(0, (|h_1|^2 + |h_2|^2) N_0)$$

Power constraint $\frac{P}{2}$

$\Rightarrow$ For given $\mathbf{h}$, capacity is $\log\left(1 + \frac{\|\mathbf{h}\|^2}{2} \mathrm{SNR}\right)$.

$$P_{out}(R) = P\left(\log\left(1 + \frac{\|\mathbf{h}\|^2}{2} \mathrm{SNR}\right) < R \right)$$

→ Compared to receive diversity (1×2), there is a power loss by a factor $\frac{1}{2}$.

($\frac{1}{L}$ in the $L \times 1$ case)

→ For 2×1, it can be shown that the Alamouti scheme is optimal in the sense that it has minimum outage probability.

(See Exercises 5.14 – 5.15 – 5.16)

→ For $L \times 1$

$$P_{out}(R) = P\left(\log\left(1 + \frac{\|\mathbf{h}\|^2}{L} \mathrm{SNR}\right) < R \right)$$

For reasonably small values of outage prob., it is optimal to use all antennas. For very high outage prob., using fewer antennas with uniform power allocation may be optimal.

Another example: Repetition code.

$$P_{out}(R) = P_r\left[\frac{1}{L}\log\left(1+\|\underline{h}\|^2\,SNR\right) < R\right]$$

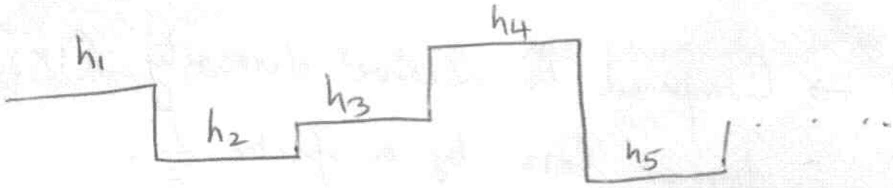
$\frac{1}{L}$ because we are sending only one symbol in L transmissions.

$\boxed{\text{Lecture 36:}}$ (3 Nov 2008)

Block fading / Parallel channels:

Suppose we have block fading model
- channel is constant over one block
- i.i.d. from block to block.

Eg: Coding over several coherence intervals.



Suppose we code over L such blocks.
each with T_c symbols (coh. time)

$$y_\ell[m] = h_\ell\, x_\ell[m] + w_\ell[m] \qquad m = 1, \ldots, T_c, \quad \ell = 1, \ldots, L.$$

Av. power constraint P for each block
$\Rightarrow$ power constraint LP over L blocks.

This can be modeled as a set of L parallel channels with total power constraint LP.

* Water-filling power allocation is optimal if the transmitter knows the channel.

* If equal power is allocated to each block, a total rate of $\sum_{\ell=1}^{L} \log(1+|h_\ell|^2 SNR)$ can be achieved over the L blocks by choosing a capacity-achieving AWGN code with rate $\log(1+|h_\ell|^2 SNR)$ for the ℓ^{th} block.

$\Rightarrow$ Average rate of transmission $= \frac{1}{L}\sum_{\ell=1}^{L} \log(1+|h_\ell|^2 SNR)$.

* It turns out that this average rate can also be achieved using a single code over the L blocks without transmitter channel knowledge.

Therefore, outage probability if we code over L blocks is now

$$P_{out}(R) = P_r\left[\frac{1}{L}\sum_{\ell=1}^{L} \log(1+|h_\ell|^2 SNR) < R\right].$$

* This result is also applicable if we get L parallel channels from frequency diversity instead of time diversity above.

* Note: Coding across parallel channels is necessary when the h_ℓ's are not known at the transmitter.

Fast fading channel:

Consider the block-fading model described in the previous section. What happens when L is large? i.e, what happens when coding over a large number of independently fading blocks?

(Law of large numbers) $$\frac{1}{L}\sum_{l=1}^{L} \log(1+|h_l|^2 SNR) \longrightarrow E\left[\log(1+|h|^2 SNR)\right]$$

A reliable rate (with arbitrarily low prob. of error) of communication of $E\left[\log(1+|h|^2 SNR)\right]$ can be achieved.

This is the capacity $C = E\left[\log(1+|h|^2 SNR)\right]$.

The above rate can be achieved even if h_l's are not i.i.d but from an ergodic process.

C is called the ergodic capacity.

* $C \leq C_{awgn}$

i.e. $$E\left[\log(1+|h|^2 SNR)\right] \leq \log(1+SNR)$$

By Jensen's inequality

$$E\left[\log(1+|h|^2 SNR)\right] \leq \log\left[E\left[1+|h|^2 SNR\right]\right]$$
$$= \log(1+E[|h|^2]SNR)$$
$$(E[|h|^2]=1) \quad = \log(1+SNR)$$

At low SNR

$$C \approx E\left[|h|^2 SNR\right] \log_2 e = SNR \log_2 e \approx C_{awgn}$$

$$\underline{C \approx C_{awgn}}$$

* At high SNR

$$C \approx E\left[\log(|h|^2 SNR)\right]$$
$$= E\left[\log |h|^2\right] + \log SNR$$
$$\approx E\left[\log |h|^2\right] + C_{awgn}.$$

$$C_{awgn} - C \approx \text{constant} = -E\left[\log |h|^2\right]$$
$$\approx 0.83 \text{ bits/s/Hz}$$

(for Rayleigh fading model.)

Power/rate control using

Transmitter side information / channel knowledge at the transmitter

Slow fading: Power control to achieve channel inversion & reduce outage probability.

$$P\left(\log\left(1 + \frac{P^*(h)|h|^2}{N_0}\right) < R\right)$$

$P^*(h)$: optimal power allocation.

- More power when channel is bad
  Less power when channel is good.

- Outage can be significantly reduced.

Fast fading: Water-filling power allocation

$$C_{erg} = E\left[\log\left(1 + \frac{P^*(h)|h|^2}{N_0}\right)\right]$$

- More power when channel is good
- Less power when channel is bad

- Improves C_{erg} compared to no CSIT.

  At high SNR
  - improvement negligible.

  At low SNR
  - significant improvement
  - Infact, C_{erg} with CSIT can be greater than C_{awgn} at very low SNR

Opportunistic communication → Communicate only when channel realization is well above average channel.

Frequency-selective fading channels:

With OFDM, a freq-sel. channel is transformed into a set of parallel flat fading channels. This has already been discussed.