$\boxed{\text{Lecture 32:}}$ (24 Oct 2008)

Let us study how C depends on $\bar{P}$ (Power)
& W (bandwidth).

(1) Suppose W is fixed.

As we increase $\bar{P}$, C increases.

But how?

$$C = W \log_2\left(1 + \frac{\bar{P}}{N_0 W}\right) \text{ bits/s.}$$

For small $\frac{\bar{P}}{N_0 W}$, $C \approx W \frac{\bar{P}}{N_0 W} \log_2 e$ bits/s

($\log(1+x) \approx x$ for small x)

$\Rightarrow$ Capacity increases linearly with $\bar{P}$
if $\bar{P}$ doubles, capacity doubles.

For large $\frac{\bar{P}}{N_0 W}$, $C \approx W \log_2 \frac{\bar{P}}{N_0 W}$ bits/s

($\log(1+x) \approx \log x$ for large x)

$\Rightarrow$ Capacity increases logarithmically with $\bar{P}$
if $\bar{P}$ doubles, capacity increases by 1 bit/s/Hz
(W bits/s)

Concave fn $\rightarrow$ diminishing returns.

(2) Suppose $\bar{P}$ is fixed. What happens if we increase W?

- No of degrees of freedom increases
- SNR per DOF decreases.

→ C is an increasing concave fn. of W.

→ For small W, SNR per deg. of freedom is high.
Increasing W rapidly increases C since the increase in DOF more than compensates for the decrease in SNR per DOF.

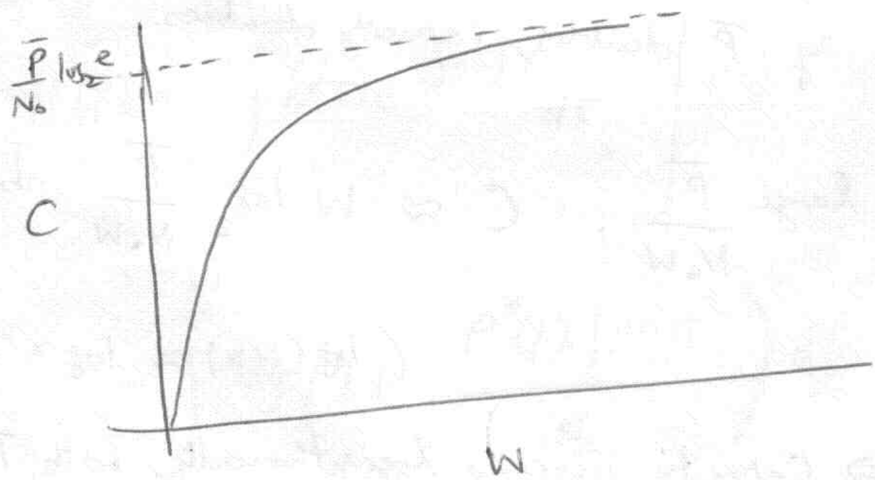
→ For large W, SNR per deg. of freedom is low.
Increasing W has a very small impact on C since reduction in SNR per DOF also decrease C significantly.

$$W \log\left(1+\frac{\bar{P}}{N_0 W}\right) \approx W \frac{\bar{P}}{N_0 W}\log_2 e \approx \frac{\bar{P}}{N_0}\log_2 e.$$

($\log(1+x) \approx x$ for small x)

This is the power-limited region. C depends mainly on $\bar{P}$.

→ As $W \to \infty$, $C(W) \to \frac{\bar{P}}{N_0}\log_2 e$ bits/s.



Even if bandwidth is not constrained, the capacity of the AWGN channel with limited power is finite.

→ At a given power level $\bar{P}$,

min. reqd. E_b (energy per bit) $= \dfrac{\bar{P}}{C_{awgn}(\bar{P}, W)}$

(for rate $R = C_{awgn}(\bar{P}, W)$)
(Use R = capacity to minimize E_b)

$$= \frac{\bar{P}}{W \log\left(1+\frac{\bar{P}}{N_0 W}\right)}.$$

$$\left(\frac{E_b}{N_0}\right)_{min} = \frac{\frac{\bar{P}}{N_0 W}}{\log_2\left(1+\frac{\bar{P}}{N_0 W}\right)}$$

→ As $\bar{P} \to 0$, we have $\left(\frac{E_b}{N_0}\right)_{min} \longrightarrow \frac{1}{\log_2 e} = -1.59\text{ dB}$.

($\left(\frac{E_b}{N_0}\right)_{min}$ with be higher in the finite BW regime)

→ To achieve $\left(\frac{E_b}{N_0}\right)_{min} = -1.59\text{ dB}$, i.e., energy efficiency,

for a fixed W, data rate in bits/s → 0

& delay increases.

Energy is spread over a long time interval.

for infinite W, power is spread over a large bandwidth.

Orthogonal codes achieve capacity in the infinite BW regime.

Linear time-invariant Gaussian channels:

Three simple examples

- Capacity can be easily computed
- Optimal codes can be constructed from optimal codes for the AWGN channel.
- Time-invariant
- channel known at the transmitter and at the receiver.
  ↳ (for examples 2 & 3)

(1) SIMO channel.

$$y_l[m] = h_l x[m] + w_l[m] \qquad l = 1, \dots, L.$$

$w_\ell[m] \sim \mathcal{CN}(0, N_0)$ i.i.d.

h_ℓ : fixed complex channel gain from tx. antenna to ℓ^{th} rx. a

$$\rightarrow \quad \tilde{y}[m] = \underline{h}^H \underline{y}[m] \quad \text{is a sufficient statistic} \rightarrow \text{(A)}$$

$$\underline{y}[m] = \begin{bmatrix} y_1[m] \\ y_2[m] \\ \vdots \\ y_L[m] \end{bmatrix} \qquad \underline{h} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{bmatrix}$$

$$\tilde{y}[m] = \|\underline{h}\|^2 x[m] + \underline{h}^H \underline{w}[m]$$

$$\Rightarrow \boxed{\tilde{y}[m] = \|\underline{h}\|^2 x[m] + \underbrace{\tilde{w}[m]}_{\mathcal{CN}(0, N_0\|\underline{h}\|^2)}}$$

AWGN channel with SNR (received) $\dfrac{P\|\underline{h}\|^2}{N_0}$

where P is the average transmit energy per symbol.

$$C = \log\left(1 + \frac{P\|\underline{h}\|^2}{N_0}\right) \text{ bits/s/Hz}.$$

bits/transmission.

Multiple receive antennas increase the received SNR, & provide a power gain.

The above combining scheme in (A) is called <u>receive beamforming</u>.

Lecture 33 : (28 Oct 2008)

(2) MISO channel.

$$y[m] = \underline{h}^H \underline{x}[m] + \underbrace{w[m]}_{\mathcal{CN}(0, N_0) \text{ i.i.d.}}$$

$\underline{h} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{bmatrix}$ fixed channel gain vector

Total Power constraint P across the L antennas.

→ Send information in the direction of $\underline{h}$

Information in orthogonal directions (to $\underline{h}$) will be nulled by the channel anyway.

$$\Rightarrow \text{Set } \underline{x}[m] = \frac{\underline{h}}{\|\underline{h}\|} \tilde{x}[m] \quad \text{— (B)}$$

MISO channel is reduced to the foll. scalar AWGN channel

$$y[m] = \|\underline{h}\| \tilde{x}[m] + w[m]. \quad \text{— (C)}$$

P is the power constraint on $\tilde{x}[m]$.

MISO with CSIT+CSIR ≡ SIMO with CSIR ←
$$C = \log\left(1 + \frac{P\|\underline{h}\|^2}{N_0}\right) \text{ bits/use.}$$

The above scheme (B) can be shown to be the best scheme. (Exercise 5.11)

{ [Proof by contradiction] If it is possible to beat this scheme, it is possible to construct a code for the scalar AWGN channel in (C) that achieves rate > capacity. }

(3) Frequency-selective channel

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell \, x[m-\ell] + w[m]$$

→ L-tap time-invariant channel

→ Known at transmitter & receiver.

The frequency-selective channel can be converted into N_c independent parallel channels (subcarriers) by adding a CP of length $L-1$ to a data vector of length N_c.

For the i^{th} OFDM block above, we can write

$$\tilde{y}_n[i] = \tilde{h}_n \tilde{d}_n[i] + \tilde{w}_n[i] \quad \text{for } n = 0, 1, \ldots, N_c - 1.$$

$$\underline{\tilde{d}}[i] = \begin{bmatrix} \tilde{d}_0[i] \\ \tilde{d}_1[i] \\ \vdots \\ \tilde{d}_{N_c-1}[i] \end{bmatrix} \qquad \underline{\tilde{w}}[i] = \begin{bmatrix} \tilde{w}_0[i] \\ \tilde{w}_1[i] \\ \vdots \\ \tilde{w}_{N_c-1}[i] \end{bmatrix} \qquad \underline{\tilde{y}}[i] = \begin{bmatrix} \tilde{y}_0[i] \\ \tilde{y}_1[i] \\ \vdots \\ \tilde{y}_{N_c-1}[i] \end{bmatrix}$$

$\underline{\tilde{h}}$ DFT of the channel, length of DFT $= N_c$.

As $N_c \to \infty$, CP overhead $\dfrac{L-1}{L-1+N_c} \to 0$.

$\Rightarrow$ Capacity of original freq. selective channel

= Capacity of transformed $\|^{l}$ channels as $N_c \to \infty$.

We have a set of N_c parallel channels with an overall power constraint P. Capacity is achieved by capacity-achieving AWGN codes for each of these channels + optimal power allocation across channels.

$$C_{N_c} = \max_{P_0, P_1, \ldots, P_{N_c-1}} \sum_{n=0}^{N_c-1} \log\left(1 + \frac{P_n |\tilde{h}_n|^2}{N_0}\right)$$

Subject to $\sum_{n=0}^{N_c-1} P_n \leq P$ (Power constraint is $N_c P$ in text)

Optimal power allocation:

$$\sum_{n=0}^{N_c-1} \log\left(1+\frac{P_n|\tilde{h}_n|^2}{N_0}\right)$$ is jointly concave in the powers.

$\Rightarrow$ Use Lagrangian method (dual problem)

$$\mathcal{L} = \sum_{n=0}^{N_c-1} \log\left(1+\frac{P_n|\tilde{h}_n|^2}{N_0}\right) + \lambda\left(\sum_{n=0}^{N_c-1} P_n - P\right)$$

KKT conditions:

$$\frac{\partial \mathcal{L}}{\partial P_n} = \begin{cases} 0 & \text{if } P_n>0 \text{, for } n=0,1,\dots,N_c-1 \\ \le 0 & \text{if } P_n = 0 \end{cases}$$

$$\Rightarrow \quad P_n^* = \left(-\frac{1}{\lambda} - \frac{N_0}{|\tilde{h}_n|^2}\right)^+ = \left(\nu - \frac{N_0}{|\tilde{h}_n|^2}\right)^+.$$

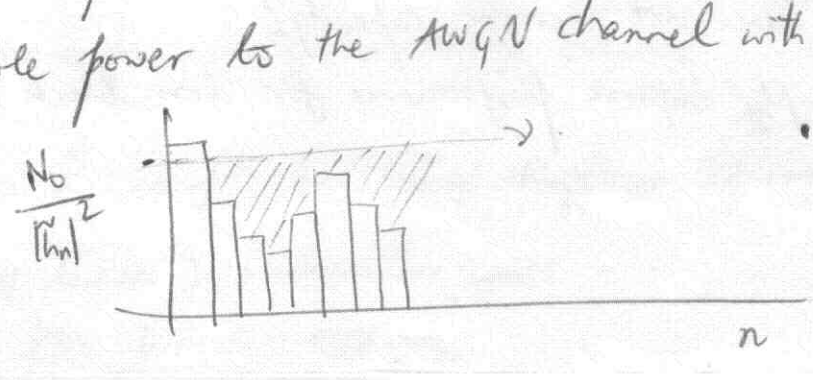
$\left(\frac{-1}{\lambda} \triangleq \nu\right)$

Choose λ such that

$$\sum_{n=0}^{N_c-1} P_n^* = P$$ (Power constraint is $N_c P$ in text)

The above solution can be interpreted as a "water-filling" power allocation.

(Note: The optimal strategy is not to allocate all available power to the AWGN channel with highest SNR)



$$\tilde{h}_n = \frac{1}{\sqrt{N_c}} \sum_{l=0}^{L-1} h_l \, e^{-j\frac{2\pi l n}{N_c}}$$

This is $H(f) = \sum_{l=0}^{N_c-1} h_l \, e^{-j\frac{2\pi l f}{W}}$ $\quad f \in [0, W]$

evaluated at $f = \frac{nW}{N_c}$

scaled by $\frac{1}{\sqrt{N_c}}$.

As $N_c \to \infty$, $\quad \tilde{h}_n = \frac{H(\frac{nW}{N_c})}{\sqrt{N_c}}$

$$P^*(f) = \left(\nu - \frac{N_0}{|H(f)|^2}\right)^+$$

$$\int_0^W P^*(f)\,df = P$$

$$\left[H(f) = \int_{-\infty}^{\infty} h(\tau) e^{-j2\pi f \tau} d\tau \right]$$

$$C = \lim_{N_c \to \infty} \frac{C_{N_c}}{N_c} \cdot W$$

$$C = \int_0^W \log\left(1 + \frac{P^*(f)\,|H(f)|^2}{N_0}\right) df$$

Note

Coding across subcarriers:

- Does not increase capacity?
- May improve performance for low block lengths.

Optimal power allocation across parallel Gaussian channels

$$C = \max_{\{P_n\}} \sum_{n=0}^{N_c-1} \log\left(1+\frac{P_n|\tilde{h}_n|^2}{N_0}\right) = \min_{\{P_n\}} \sum_{n=0}^{N_c-1} -\log\left(1+\frac{P_n|\tilde{h}_n|^2}{N_0}\right)$$

subject to $\sum_{n=0}^{N_c-1} P_n \le P$

$P_n \geq 0 \quad n=0,1,\dots,N_c-1.$

Subject to $\sum_{n=0}^{N_c-1} P_n \le P$

$-P_n \le 0$

(Convex optimization problem in standard form)

$\min_{x} f_0(x)$

subject to $f_i(x) \le 0 \quad i=1,\dots,m$

$h_i(x) = 0 \quad i=1,\dots,p$

where f_0, f_i, h_i are convex fns.

Define

$$\mathcal{L} = -\sum_{n=0}^{N_c-1} \log\left(1+\frac{P_n|\tilde{h}_n|^2}{N_0}\right) + \lambda\left(\sum_{n=0}^{N_c-1} P_n - P\right) + \nu_n(-P_n)$$

The optimal $\{P_n^*\}$ (x^*) satisfy the KKT conditions

(See p. 244–245, Boyd & Vandenberghe, "Convex optimization", Cambridge Univ. Press)

$f_i(\underline{x}^*) \le 0 \quad i=1,2,\dots,m$ — (1)

$h_i(x^*) = 0 \quad i=1,2,\dots,p$ — (2)

} ($\{P_n^*\}$ (or x^*) is feasible).

$\lambda_i \geq 0 \quad i=1,2,\dots,m$ — (3)

$\lambda_i f_i(x^*) = 0 \quad i=1,2,\dots,m.$ — (4)

$\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i \nabla h_i(x^*) = 0.$ — (5)

These conditions translate to the following eqns. in our problem.

(5) $\Rightarrow \dfrac{\partial \mathcal{L}}{\partial P_n} = 0$ for $n=0,1,\dots,N_c-1$ at $P_n = P_n^*$

$$-\frac{\frac{|\tilde{h}_n|^2}{N_0}}{1+\frac{P_n^*|\tilde{h}_n|^2}{N_0}} + \lambda - \nu_n = 0 \Rightarrow -\frac{1}{P_n^* + \frac{N_0}{|\tilde{h}_n|^2}} + \lambda - \nu_n = 0 \quad \text{— (A)}$$

$n=0,1,\dots,N_c-1.$

(4) $\Rightarrow \lambda\left(\sum_{n=0}^{N_c-1} P_n^* - P\right) = 0$ i.e., either $\lambda = 0$ or $\sum_{n=0}^{N_c-1} P_n^* = P$. — (B)

(4) $\Rightarrow \nu_n P_n^* = 0$ for $n=0,1,\dots,N_c-1$ — (C)

(3) $\Rightarrow \lambda \geq 0,\ \nu_n \geq 0.$ — (D)

(2) No equality constraints in our problem.

(1) $\Rightarrow \sum_{n=0}^{N_c-1} P_n^* \leq P$, $P_n^* \geq 0$. — (E)

(A) $\Rightarrow \gamma_n = \lambda - \dfrac{1}{P_n^* + \frac{N_0}{|\tilde{h}_n|^2}}$.

Using this in (C) gives $\left(\lambda - \dfrac{1}{P_n^* + \frac{N_0}{|\tilde{h}_n|^2}}\right) P_n^* = 0$. $n = 0, 1, \cdots, N_c - 1$

$\Rightarrow$ either $P_n^* = 0$ (or) $\lambda = \dfrac{1}{P_n^* + \frac{N_0}{|\tilde{h}_n|^2}}$ i.e. $P_n^* = \dfrac{1}{\lambda} - \dfrac{N_0}{|\tilde{h}_n|^2}$.

We also need $P_n^* \geq 0$. (E).

$\Rightarrow$ If $\dfrac{1}{\lambda} - \dfrac{N_0}{|\tilde{h}_n|^2} > 0$, $P_n^* = \dfrac{1}{\lambda} - \dfrac{N_0}{|\tilde{h}_n|^2}$

else, $P_n^* = 0$.

i.e. $\boxed{P_n^* = \max\left(0, \dfrac{1}{\lambda} - \dfrac{N_0}{|\tilde{h}_n|^2}\right) = \left(\dfrac{1}{\lambda} - \dfrac{N_0}{|\tilde{h}_n|^2}\right)^+}$

Using all the power gives better capacity here.

$\rightarrow \boxed{\sum_{n=0}^{N_c-1} P_n^* = P} \longrightarrow \lambda$ is found by solving this equation. (also $\Rightarrow$ (B))

Optimal power allocation solution given by these 2 equations.

Remarks

- Interpretation as a "water-filling" solution
- Start with channel with best SNR, Use more channels as more power becomes available. More power for channels with higher SNR (higher $\frac{|\tilde{h}_n|^2}{N_0}$).
- When $P \to \infty$, uniform power allocation is also good (close to optimal solution).