Lecture 12. (25/8/2008)

Rotation code:

* We saw that diversity L can be achieved using a repetition code. However, only one BPSK symbol was sent in $2(L)$ [in general] intervals.

* We can send 2 bits in 2 symbols using QPSK by doing the same repetition on both real & imaginary parts.

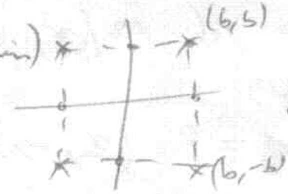
* How can we send 2 bits in 1 symbol and still have diversity 2?

  Using a 16-symbol constellation and repeating it twice.

  i.e. 4-PAM in each dimension.

Av. energy $= \frac{20b^2}{4} = 5b^2$.

QPSK (for same d_{min})



Av. energy $= 2b^2$.

For same energy per bit, d_{min} reduces by a factor of $\sqrt{5}$.

* Using the rotation code, we can transmit 2 bits/symbol and still get diversity 2.

  We will see what to do in each dimension, i.e., 2 bits over $L=2$ transmissions. (Overall, we have 4 bits in 2 transmissions).

* Consider $u_1 = \pm a$ & $u_2 = \pm a$.

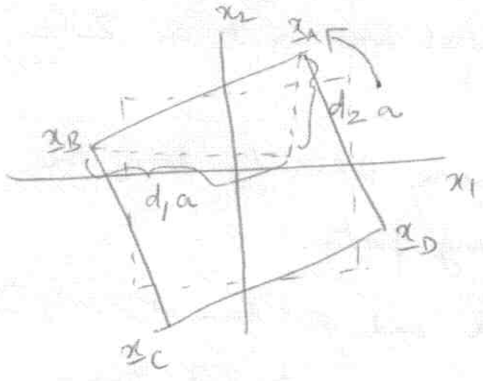
  Over the 2 symbol intervals, transmit $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

  given by $\underline{x} = R \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ where $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

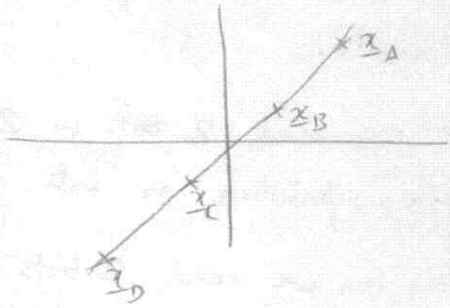
  the rotation matrix.

4 possible length-2 codewords over 2 intervals are :

$$\underline{x}_A = R\begin{pmatrix} a \\ a \end{pmatrix}, \quad \underline{x}_B = R\begin{pmatrix} -a \\ a \end{pmatrix}, \quad \underline{x}_C = R\begin{pmatrix} -a \\ -a \end{pmatrix}, \quad \underline{x}_D = R\begin{pmatrix} a \\ -a \end{pmatrix}.$$



If we use repetition, we use 4-PAM for same rate



This does not use the available $\boxed{\text{degrees of freedom.}}$

<u>Error probability for rotation code:</u>

$$\Pr(\text{error}) = \Pr(\text{error} \,/\, \underline{x}_A \text{ is transmitted})$$

$$= \underbrace{\Pr(\underline{x}_A \to \underline{x}_B)}_{} + \Pr(\underline{x}_A \to \underline{x}_C) + P(\underline{x}_A \to \underline{x}_D)$$

Prob. of decoding $\underline{x}_B$ when $\underline{x}_A$ is transmitted.

$$\leq \underbrace{\mathbb{P}(\underline{x}_A \to \underline{x}_B)}_{} + \mathbb{P}(\underline{x}_A \to \underline{x}_C) + \mathbb{P}(\underline{x}_A \to \underline{x}_D)$$

Prob. of decoding $\underline{x}_B$ when $\underline{x}_A$ is transmitted given a constellation with only 2 symbols $\underline{x}_A$ & $\underline{x}_B$

Let h_1, h_2 be the channel fades during intervals 1 and 2.

$$\Pr(\underline{x}_A \to \underline{x}_B \mid h_1, h_2) = Q\left(\frac{d_{A,B}}{2\sqrt{N_0/2}}\right)$$

$$= Q\left(\frac{\left\| \begin{pmatrix} h_1 x_{A_1} \\ h_2 x_{A_2} \end{pmatrix} - \begin{pmatrix} h_1 x_{B_1} \\ h_2 x_{B_2} \end{pmatrix} \right\|}{2\sqrt{N_0/2}}\right)$$

$$\left(SNR = \frac{a^2}{N_0}\right)$$

$$= Q\left(\sqrt{\frac{SNR\left(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2\right)}{2}}\right)$$

$$P(\underline{x}_A \to \underline{x}_B \mid h_1, h_2) \leq Q\left(\sqrt{\frac{SNR\left(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2\right)}{2}}\right)$$

$$\left(Q(x) \leq e^{-x^2/2}\right) \qquad \leq \exp\left(-\frac{SNR\left(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2\right)}{4}\right)$$

Averaging with respect to h_1 and h_2 under the Rayleigh-fading model

$$P(\underline{x}_A \to \underline{x}_B) \leq E_{h_1,h_2}\left[\exp\left(\frac{-SNR\left(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2\right)}{4}\right)\right]$$

$$= E_{h_1}\left[\exp\left(\frac{-SNR\,|h_1|^2|d_1|^2}{4}\right)\right] E_{h_2}\left[\exp\left(\frac{-SNR\,|h_2|^2|d_2|^2}{4}\right)\right]$$

$$\left[\begin{array}{l} X_1 \triangleq |h_1|^2 \sim \text{exponential} \quad f_{X_1}(x_1) = e^{-x_1} \\ E\left[e^{sX}\right] = \dfrac{1}{1-s} \quad \text{for } s < 1. \end{array}\right]$$

$$= \left[\frac{1}{1 + \left(SNR\,|d_1|^2/4\right)}\right]\left[\frac{1}{1 + \left(SNR\,|d_2|^2/4\right)}\right]$$

At high SNR,

$$P_r(\underline{x}_A \to \underline{x}_B) \leq \frac{16}{|d_1 d_2|^2} SNR^{-2}$$

Let $\delta_{AB} \triangleq |d_1 d_2|^2$ Squared product distance between $\underline{x}_A$ & $\underline{x}_B$ when average energy is normalized to 1 per symbol time

[(Product of the distance along each coordinate)^2]

Similarly, we can bound $P(\underline{x}_A \to \underline{x}_C)$ & $P(\underline{x}_A \to \underline{x}_D)$.

Therefore,

$$Pr(\text{error}) \leq 16\left(\frac{1}{\delta_{AB}} + \frac{1}{\delta_{AC}} + \frac{1}{\delta_{AD}}\right) SNR^{-2}$$

will determine coding gain

$$\leq \frac{48}{\min\limits_{j=B,C,D} \delta_{Aj}} SNR^{-2}.$$

Therefore, as long as $\delta_{ij} > 0$ for all i, j, we get a diversity gain of 2.

$\min\limits_{j} \delta_{Aj}$ determines performance $\to$ larger min. product distance implies more coding gain.

* Tightness of upper bound at high SNR (See problem 3.7).
* The rotation angle can be chosen to maximize coding gain. $\left(\theta^* = \frac{1}{2}\tan^{-1}(2),\ \min \delta_{ij} = \frac{16}{5}\right)$ (exercise)

$\boxed{\text{Lecture 13.}}$ 26/8/2008

* <u>Interpretation on why product distance is important</u>

$\underline{x}_A$ is confused with $\underline{x}_B$ if sq. Euclidean distance $|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2$ is of the order of $\frac{1}{SNR}$.

This happens when both $|h_1|^2|d_1|^2$ and $|h_2|^2|d_2|^2$ are of the order of $\frac{1}{SNR}$. This happens with probability

$$\left(\frac{1}{|d_1|^2 SNR}\right)\left(\frac{1}{|d_2|^2 SNR}\right) = \frac{1}{|d_1|^2|d_2|^2} SNR^{-2}.$$

* Coding gain over repetition code $\approx$ 3.5 dB (factor of $\sqrt{5}$ in d_{min}) (exercise)

* Rotation code has better product distance than the repetition code. The codewords are packed in a 2-D space rather than on a 1-D line as in the repetition code. $\Rightarrow$ Better use of the available $\boxed{\text{degrees of freedom}}$

Generalization to any time diversity code:

Let $\underline{x}_1, \ldots, \underline{x}_M$ (L×1 vectors) be the codewords of a time diversity code over L transmissions (block length L).

(Ideal i.i.d channel model)

$$y_\ell = h_\ell x_\ell + w_\ell \quad \text{for } \ell = 1, 2, \ldots, L$$

$$h_\ell \sim \text{i.i.d } CN(0,1)$$

$$\frac{1}{ML} \sum_{i=1}^{M} \|\underline{x}_i\|^2 = 1 \quad \text{(Power constraint) Normalization.}$$

$$P_e \le \frac{1}{M} \sum_{i \ne j} P(\underline{x}_i \to \underline{x}_j) \quad \text{(Symmetry assumption?)}$$

$$P(\underline{x}_i \to \underline{x}_j) \le \prod_{\ell=1}^{L} \frac{1}{1 + \left(SNR\,|x_{i\ell} - x_{j\ell}|^2 / 4\right)}$$

where $x_{i\ell}$ is the ℓ^{th} component of $\underline{x}_i$, $SNR = \frac{1}{N_0}$.

Let L_{ij} be the number of components on which codewords $\underline{x}_i$ & $\underline{x}_j$ differ. Diversity gain of the code is

$$\boxed{\min_{i \ne j} L_{ij}}.$$

If $L_{ij} = L$ for all $i \neq j$, then the code achieves full diversity L

$$P_e \leq \frac{4^L}{M} \sum_{i \neq j} \frac{1}{\delta_{ij}} SNR^{-L} \leq \frac{4^L (M-1)}{\min_{i \neq j} \delta_{ij}} SNR^{-L}$$

where $\delta_{ij} = \prod_{l=1}^{L} |x_{il} - x_{jl}|^2$ (Squared product distance between $\underline{x}_i$ & $\underline{x}_j$).

* Binary linear block code + ideal interleaving

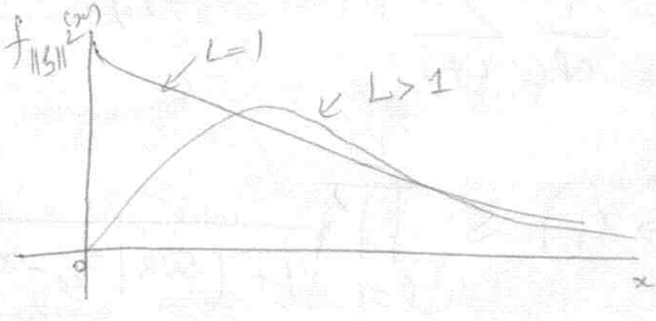
$\Rightarrow$ Diversity gain = Minimum Hamming Distance (Minimum weight)

Binary convolutional code $\Rightarrow$ Diversity gain = Free distance.

Recap: - Time diversity using coding + interleaving

- Decoding delay

- PDF of $\|\underline{h}\|^2$ for diff. values of L.



(Plot this as an exercise)

(See Fig 3.7 in the book (p63))