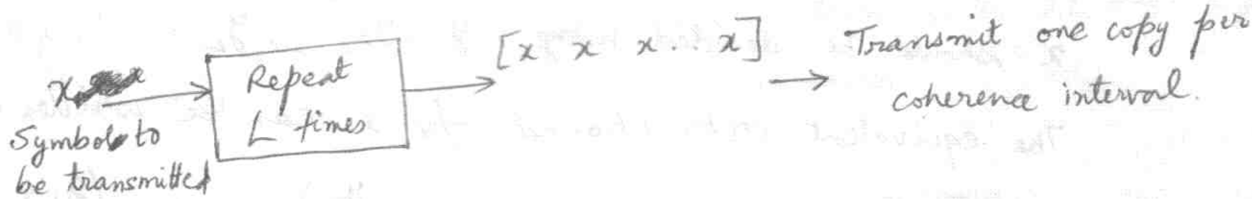
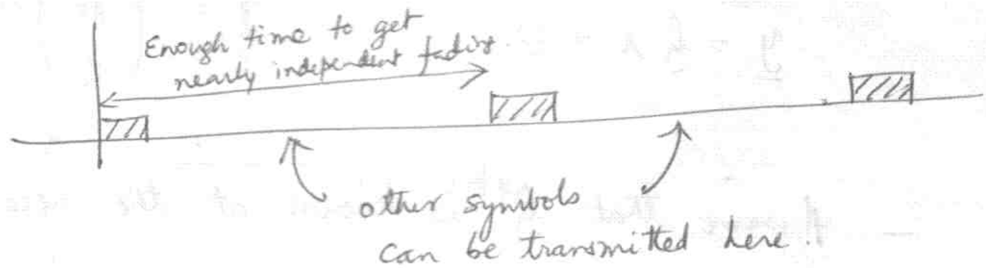
Lecture 11: (22/8/2008)

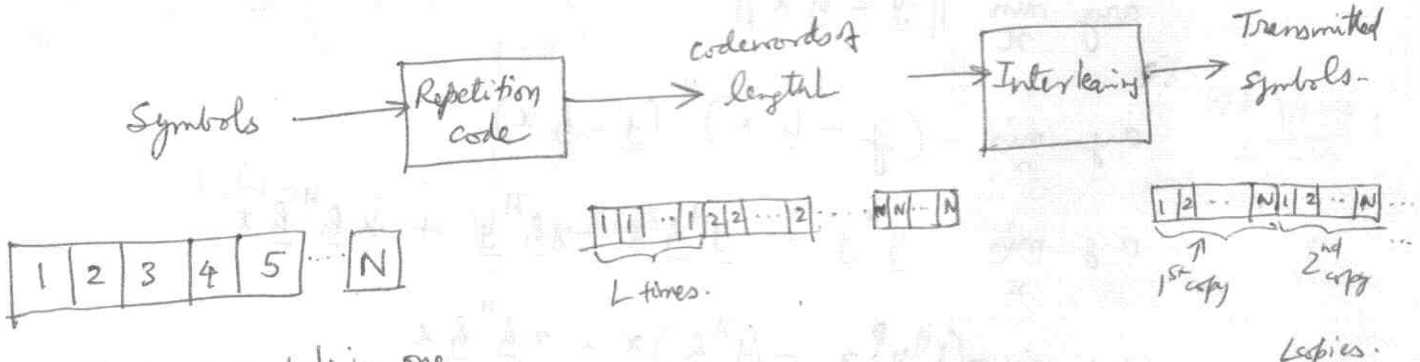
Diversity:

Time Diversity: To achieve ~~[illegible]~~ diversity, symbols have to be transmitted through independent or nearly independent channels. Channel coherence time is of the order of a few ms, which corresponds to tens or hundreds of symbols (depending on the data rate). Here is one way to achieve time diversity.





This leads to the following model.



N : Number of symbols in one coherence interval

A more general method could be one where the repetition code is replaced by ~~a~~ more effective error control code.

* What is the optimal detection strategy to minimize symbol error rate for the repetition code case?

— Each symbol x is transmitted L times.

Assuming that each transmission encounters an independent Rayleigh flat fading channel we will receive $y_1, y_2, \cdots, y_L$ given by

$$y_i = h_i x + w_i$$

where h_i's are $\mathcal{CN}(0,1)$ and independent

w_i's are $\mathcal{CN}(0, N_0)$ and independent.

x should be decoded using $y_1, y_2, \cdots, y_L$.

The equivalent vector channel for x can be written as

$$\underline{y} = \underline{h} x + \underline{w} \qquad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{pmatrix}, \quad \underline{h} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{pmatrix}, \quad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{pmatrix}$$

— Assume that $\underline{h}$ is known at the receiver.

Optimal detection rule:

$$\arg\min_x \|\underline{y} - \underline{h} x\|^2$$

$$\arg\min_x (\underline{y} - \underline{h} x)^H (\underline{y} - \underline{h} x)$$

$$\arg\min_x \underline{y}^H \underline{y} - \underline{y}^H \underline{h} x - x \underline{h}^H \underline{y} + x \underline{h}^H \underline{h} x$$

$$\arg\min_x -(\underline{h}^H \underline{y})^* x - (\underline{y}^H \underline{h}) x + x \underline{h}^H \underline{h} x$$

$\underline{h}^H \underline{y}$ is a sufficient statistic.

$$r = \underline{h}^H \underline{y} = \|\underline{h}\|^2 x + \underbrace{\underline{h}^H \underline{w}}_{\mathcal{CN}(0, \|\underline{h}\|^2 N_0)}$$

Consider BPSK, $x = \pm a$.

Detection rule: $\text{Re}(\underline{h}^H \underline{y}) > 0 \Rightarrow \hat{x} = a$

else $\hat{x} = -a$.

$$P(\text{error} \mid \underline{h}) = Q\left(\frac{2\|\underline{h}\|^2 a}{2\sqrt{\|\underline{h}\|^2 N_0/2}}\right) = Q\left(\sqrt{\frac{2\|\underline{h}\|^2 a^2}{N_0}}\right)$$

$$= Q\left(\sqrt{2\|\underline{h}\|^2 \text{SNR}}\right)$$

$$\left(\text{SNR} = \frac{a^2}{N_0}\right).$$

(Proof: Exercise)

$$\|\underline{h}\|^2 = \sum_{l=1}^{L} |h_l|^2 \sim \text{chi-square distribution with } 2L \text{ degrees of freedom}$$

$$X = \|\underline{h}\|^2 \sim f_X(x) = \frac{1}{(L-1)!} x^{L-1} e^{-x} \qquad x \geq 0.$$

$$\Pr(\text{error}) = \int_0^\infty Q\sqrt{2x\,\text{SNR}}\; f_X(x)\,dx.$$

(Proof: exercise)

$$= \left(\frac{1-\mu}{2}\right)^L \sum_{l=0}^{L-1} \binom{L-1+l}{l} \left(\frac{1+\mu}{2}\right)^l$$

where $\mu = \sqrt{\dfrac{\text{SNR}}{1+\text{SNR}}}$.

Also

$$= \frac{1}{2}\left[1 - \mu \sum_{l=0}^{L-1} \binom{2l}{l} \left(\frac{1-\mu^2}{4}\right)^l\right]$$

At high SNR:

$$\frac{1+\mu}{2} \approx 1$$

$$\frac{1-\mu}{2} \approx \frac{1}{4\,\text{SNR}}$$

Also, $\displaystyle\sum_{l=0}^{L-1} \binom{L-1+l}{l} = \binom{2L-1}{L}$.

Therefore, at high SNR,

$$\Pr(\text{error}) \approx \binom{2L-1}{L} \frac{1}{(4\,\mathrm{SNR})^L}.$$

* Error probability decreases as the L^{th} power of SNR
  (corresponds to a slope of $-L$ in the dB/dB scale Pe vs SNR plot)

* Dominant error event :

  "Deep fade" $\quad \|\mathbf{h}\|^2 < \dfrac{1}{\mathrm{SNR}}$

$$P\left(\|\mathbf{h}\|^2 < \frac{1}{\mathrm{SNR}}\right) = \int_0^{1/\mathrm{SNR}} f_{\|\mathbf{h}\|^2}(x)\, dx$$

For small x

$$f_{\|\mathbf{h}\|^2}(x) \approx \frac{1}{(L-1)!} x^{L-1} \qquad \left(x^{L-1} e^{-x} \approx x^{L-1}\right)$$

$$P(\text{Deep fade}) \approx \int_0^{1/\mathrm{SNR}} \frac{1}{(L-1)!} x^{L-1}\, dx = \frac{1}{L!} \frac{1}{\mathrm{SNR}^L}.$$

* Basically, an error occurs when $\sum_{l=1}^{L} |h_l|^2$ is of the order of $\dfrac{1}{\mathrm{SNR}}$ or smaller. This happens when all $|h_l|^2$ are small of the order of $\dfrac{1}{\mathrm{SNR}}$.

* L is called the <u>diversity gain</u> of the system.

Lecture 12. (25/8/2005)

Rotation code:

* We saw that diversity L can be achieved using a repetition code. However, only one BPSK symbol was sent in 2 (L in general) intervals.

* We can send 2 bits in 2 symbols using QPSK by doing the same repetition on both real & imaginary parts.

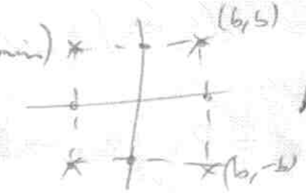
* How can we send 2 bits in 1 symbol and still have diversity 2?

  Using a 16-symbol constellation and repeating it twice.

  i.e. 4-PAM in each dimension.

$$\text{Av. energy} = \frac{20b^2}{4} = 5b^2.$$

QPSK (for same d_{min})



$$\text{Av. energy} = 2b^2.$$

For same energy per bit, d_{min} reduces by a factor of $\sqrt{5}$.

* Using the rotation code, we can transmit 2 bits/symbol and still get diversity 2.

  We will see what to do in each dimension, i.e., 2 bits over $L=2$ transmissions. (Overall, we have 4 bits in 2 transmissions).

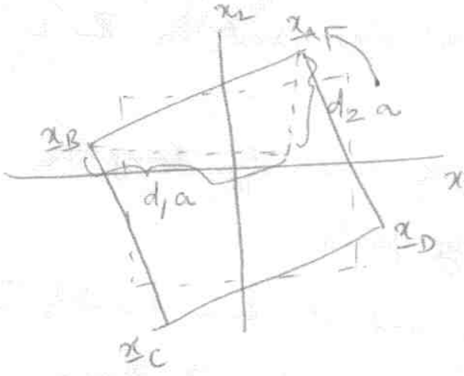
* Consider $u_1 = \pm a$ & $u_2 = \pm a$.

  Over the 2 symbol intervals, transmit $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ given by $\underline{x} = R \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ where $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

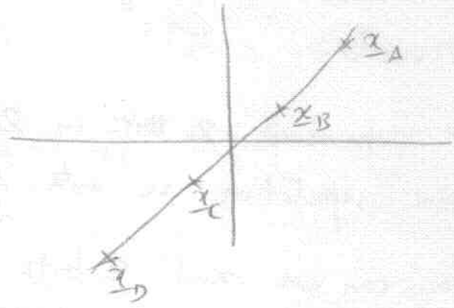
  the rotation matrix.

4 possible length-2 codewords over 2 intervals are :-

$$\underline{x}_A = R\begin{pmatrix} a \\ a \end{pmatrix}, \quad \underline{x}_B = R\begin{pmatrix} -a \\ a \end{pmatrix}, \quad \underline{x}_C = R\begin{pmatrix} -a \\ -a \end{pmatrix}, \quad \underline{x}_D = R\begin{pmatrix} a \\ -a \end{pmatrix}$$



If we use repetition, we use 4-PAM for same rate



This does not use the available $\boxed{\text{degrees of freedom.}}$

Error probability for rotation code:

$$\Pr(\text{error}) = \Pr(\text{error} \mid \underline{x}_A \text{ is transmitted})$$

$$= \underbrace{\Pr(\underline{x}_A \to \underline{x}_B)} + \Pr(\underline{x}_A \to \underline{x}_C) + P(\underline{x}_A \to \underline{x}_D)$$

Prob. of decoding $\underline{x}_B$ when $\underline{x}_A$ is transmitted.

$$\leq \underbrace{\mathbb{P}(\underline{x}_A \to \underline{x}_B)} + \mathbb{P}(\underline{x}_A \to \underline{x}_C) + \mathbb{P}(\underline{x}_A \to \underline{x}_D)$$

Prob. of decoding $\underline{x}_B$ when $\underline{x}_A$ is transmitted given a constellation with only 2 symbols $\underline{x}_A$ & $\underline{x}_B$

Let h_1, h_2 be the channel fades during intervals 1 and 2.

$$\Pr(\underline{x}_A \to \underline{x}_B \mid h_1, h_2) = Q\left(\frac{d_{A,B}}{2\sqrt{N_0/2}}\right)$$

$$= Q\left(\frac{\left\| \begin{pmatrix} h_1 x_{A_1} \\ h_2 x_{A_2} \end{pmatrix} - \begin{pmatrix} h_1 x_{B_1} \\ h_2 x_{B_2} \end{pmatrix} \right\|}{2\sqrt{N_0/2}}\right)$$

$$\left(SNR = \frac{a^2}{N_0}\right)$$

$$= Q\left(\sqrt{\frac{SNR\left(|h_1|^2 |d_1|^2 + |h_2|^2 |d_2|^2\right)}{2}}\right)$$

$$P(\underline{x}_A \to \underline{x}_B \mid h_1, h_2) \leq Q\left(\sqrt{\frac{SNR\left(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2\right)}{2}}\right)$$

$$\left(Q(x) \leq e^{-x^2/2}\right) \qquad \leq \exp\left(-\frac{SNR\left(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2\right)}{4}\right)$$

Averaging with respect to h_1 and h_2 under the Rayleigh fading model

$$P(\underline{x}_A \to \underline{x}_B) \leq E_{h_1,h_2}\left[\exp\left(\frac{-SNR\left(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2\right)}{4}\right)\right]$$

$$= E_{h_1}\left[\exp\left(\frac{-SNR\,|h_1|^2|d_1|^2}{4}\right)\right] E_{h_2}\left[\exp\left(\frac{-SNR\,|h_2|^2|d_2|^2}{4}\right)\right]$$

$$\left[\begin{array}{l} X_1 \triangleq |h_1|^2 \sim \text{exponential} \quad f_{X_1}(x_1) = e^{-x_1} \\ \qquad E\left[e^{sX}\right] = \dfrac{1}{1-s} \quad \text{for } s < 1. \end{array}\right]$$

$$= \left[\frac{1}{1 + \left(SNR|d_1|^2/4\right)}\right]\left[\frac{1}{1 + \left(SNR|d_2|^2/4\right)}\right]$$

At high SNR,

$$P_r(\underline{x}_A \to \underline{x}_B) \leq \frac{16}{|d_1 d_2|^2} SNR^{-2}$$

Let $\delta_{AB} \triangleq |d_1 d_2|^2$ Squared product distance between $\underline{x}_A$ & $\underline{x}_B$ when average energy is normalized to 1 per symbol time

$\left\{ (\text{Product of the distance along each coordinate})^2 \right\}$

Similarly, we can bound $P(\underline{x}_A \to \underline{x}_C)$ & $P(\underline{x}_A \to \underline{x}_D)$.

Therefore,

$$P_r(\text{error}) \leq 16 \left(\frac{1}{\delta_{AB}} + \frac{1}{\delta_{AC}} + \frac{1}{\delta_{AD}} \right) SNR^{-2}$$

will determine coding gain

$$\leq \frac{48}{\min\limits_{j=B,C,D} \delta_{Aj}} SNR^{-2}.$$

Therefore, as long as $\delta_{ij} > 0$ for all i, j, we get a diversity gain of 2.

$\min\limits_{j} \delta_{Aj}$ determines performance $\longrightarrow$ larger min. product distance implies more coding gain.

* Tightness of upper bound at high SNR (See problem 3.7).
* The rotation angle can be chosen to maximize coding gain. $\left(\theta^* = \frac{1}{2}\tan^{-1}(2), \; \min \delta_{ij} = \frac{16}{5}\right)$ (exercise)

$\boxed{\text{Lecture 13}}$ 26/8/2008

* <u>Interpretation on why product distance is important</u>

$\underline{x}_A$ is confused with $\underline{x}_B$ if sq. Euclidean distance $|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2$ is of the order of $\frac{1}{SNR}$.

This happens when both $|h_1|^2|d_1|^2$ and $|h_2|^2|d_2|^2$ are of the order of $\frac{1}{SNR}$. This happens with probability

$$\left(\frac{1}{|d_1|^2 SNR}\right)\left(\frac{1}{|d_2|^2 SNR}\right) = \frac{1}{|d_1|^2|d_2|^2} SNR^{-2}.$$

* Coding gain over repetition code $\approx$ 3.5 dB (exercise) (factor of $\sqrt{5}$ in d_{min})

* Rotation code has better product distance than the repetition code. The codewords are packed in a 2-D space rather than on a 1-D line as in the repetition code $\Rightarrow$ Better use of the available $\boxed{\text{degrees of freedom}}$

Generalization to any time diversity code:

Let $\underline{x}_1, \ldots, \underline{x}_M$ ($L \times 1$ vectors) be the codewords of a time diversity code over L transmissions (block length L).

$$y_\ell = h_\ell x_\ell + w_\ell \quad \text{for } \ell = 1, 2, \ldots, L$$

(Ideal i.i.d channel model)

$$h_\ell \sim \text{i.i.d } CN(0,1)$$

$$\frac{1}{ML} \sum_{i=1}^{M} \|\underline{x}_i\|^2 = 1 \quad \left(\text{Power constraint}\right) \text{ Normalization}.$$

$$P_e \leq \frac{1}{M} \sum_{i \neq j} P(\underline{x}_i \to \underline{x}_j) \quad \left(\text{Symmetry assumption?}\right)$$

$$P(\underline{x}_i \to \underline{x}_j) \leq \prod_{\ell=1}^{L} \frac{1}{1 + \left(\text{SNR} |x_{i\ell} - x_{j\ell}|^2 / 4\right)}$$

where $x_{i\ell}$ is the ℓ^{th} component of $\underline{x}_i$, $\text{SNR} = \frac{1}{N_0}$.

Let L_{ij} be the number of components on which codewords $\underline{x}_i$ & $\underline{x}_j$ differ. Diversity gain of the code is

$$\boxed{\min_{i \neq j} L_{ij}}.$$

If $L_{ij} = L$ for all $i \neq j$, then the code achieves full diversity L.

$$P_e \leq \frac{4^L}{M} \sum_{i \neq j} \frac{1}{\delta_{ij}} SNR^{-L} \leq \frac{4^L (M-1)}{\min_{i \neq j} \delta_{ij}} SNR^{-L}$$

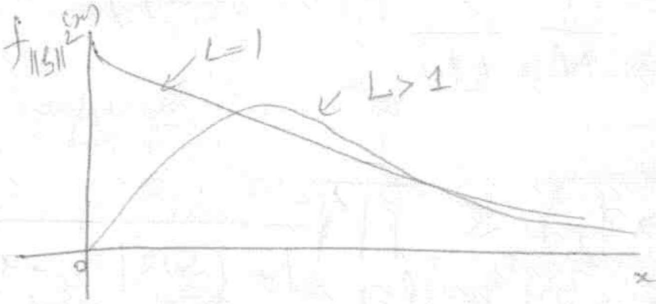
where $\delta_{ij} = \prod_{l=1}^{L} |x_{il} - x_{jl}|^2$ (Squared product distance between $\underline{x}_i$ & $\underline{x}_j$).

* Binary linear block code + ideal interleaving

$\Rightarrow$ Diversity gain = Minimum Hamming Distance (Minimum weight)

Binary convolutional code $\Rightarrow$ Diversity gain = Free distance.

Recap: - Time diversity using coding + interleaving
- Decoding delay
- PDF of $\|\underline{h}\|^2$ for diff. values of L.



(Plot this as an exercise)

(See Fig 3.7 in the book (p63))