Lecture 8 : 18 Aug 2008

Point-to-point communication:

(Orthogonal & Uncoded) Transmission over a Rayleigh fading channel:

Let us first consider communication over a simple flat fading model.

$$y[m] = h[m]\,x[m] + w[m]$$

where $h[m] \sim CN(0,1)$

$w[m] \sim CN(0, N_0)$

(Assumption on Time auto-correlation of $h[m]$ will be specified later when required.)

Case 1: $h[m]$ not known at the receiver. (Non-coherent comm.)

* BPSK signaling, $x[m] = \pm a$, fails if $h[m]$ is not known.

  Because: Phase of $y[m]$ (even in the absence of noise) is uniform in $[0, 2\pi]$ whether a or $-a$ is transmitted.

* Let us consider orthogonal signaling as follows.
  - Simple orthogonal coding
  - binary signaling over 2 samples $\rightarrow$ (m^{th} bit in $y[2m]$, $y[2m+1]$)

    Symbol A : $\underline{x}_A = \begin{pmatrix} a \\ 0 \end{pmatrix}$

    Symbol B : $\underline{x}_B = \begin{pmatrix} 0 \\ a \end{pmatrix}$.

  - To detect m^{th} bit based on $\underline{y} = \begin{pmatrix} y[2m] \\ y[2m+1] \end{pmatrix}$.

  - MAP detection minimizes prob. of error.

    If A & B are equally likely, ML detection is optimal.

Optimal detection rule:

$$\text{If} \quad f(\underline{y}/\underline{x}=\underline{x}_A) > f(\underline{y}/\underline{x}=\underline{x}_B), \quad \hat{\underline{x}} = \underline{x}_A$$

else $\hat{\underline{x}} = \underline{x}_B$.

Given $\underline{x}_A$ is transmitted, $y[2m] \sim CN(0, a^2+N_0)$

$y[2m+1] \sim CN(0, N_0)$

$y[2m], y[2m+1]$ are independent.

$$f(\underline{y}/\underline{x}=\underline{x}_A) = \frac{1}{(2\pi)\,|C_A|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{y})^H C_A^{-1}\underline{y}\right\}$$

$$C_A = \begin{bmatrix} a^2+N_0 & 0 \\ 0 & N_0 \end{bmatrix},$$

Given $\underline{x}_B$ is transmitted, $y[2m] \sim CN(0, N_0)$

$y[2m+1] \sim CN(0, a^2+N_0)$

$y[2m], y[2m+1]$ are independent.

$$f(\underline{y}/\underline{x}=\underline{x}_B) = \frac{1}{(2\pi)\,|C_B|^{1/2}} \exp\left\{-\frac{1}{2}\underline{y}^H C_B^{-1}\underline{y}\right\}$$

$$C_B = \begin{bmatrix} N_0 & 0 \\ 0 & a^2+N_0 \end{bmatrix}.$$

$$|C_A| = |C_B|$$

Optimal rule: If $\underline{y}^H C_B^{-1}\underline{y} > \underline{y}^H C_A^{-1}\underline{y}$, $\hat{\underline{x}} = \underline{x}_A$. else $\hat{\underline{x}} = \underline{x}_B$

$$\text{If} \quad \frac{1}{N_0}\|y[2m]\|^2 + \frac{1}{a^2+N_0}\|y[2m+1]\|^2 > \frac{1}{a^2+N_0}\|y[2m]\|^2 + \frac{1}{N_0}\|y[2m+1]\|^2$$

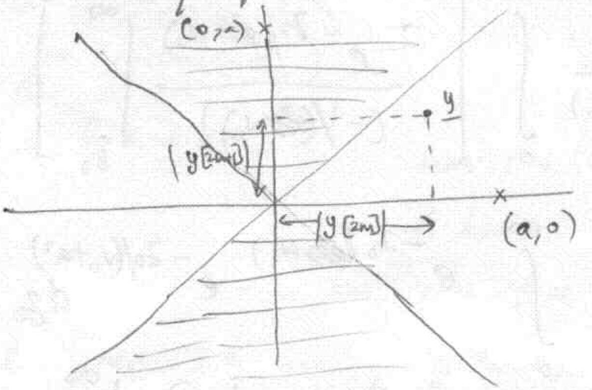
$\hat{\underline{x}} = \underline{x}_A$

$$\text{If} \quad \frac{\|y[2m]\|^2 a^2}{N_0(a^2+N_0)} - \frac{\|y[2m+1]\|^2 a^2}{N_0(a^2+N_0)} > 0 \quad \hat{\underline{x}} = \underline{x}_A$$

If $\left(|y[2m]|^2 - |y[2m+1]|^2\right) > 0$, $\hat{x} = x_A$

else $\hat{x} = x_B$

* Phases of y are not used.
* Simple energy detector / square-law detector.



* $Pr(\text{error}) = Pr(x_A)\, Pr(\text{error}/x_A) + Pr(x_B)\, Pr(\text{error}/x_B)$

$Pr(x_A) = Pr(x_B)$ (assumption)

& $Pr(\text{error}/x_A) = Pr(\text{error}/x_B)$ (symmetry)

$\Rightarrow Pr(\text{error}) = Pr(\text{error}/x_A)$

$= Pr\left(|y[2m]|^2 < |y[2m+1]|^2 / x_A\right)$

Given $x = x_A$, $y[2m] \sim CN(0, a^2+N_0) \Rightarrow |y[2m]|^2 \sim \text{expon}(a^2+N_0)$

$y[2m+1] \sim CN(0, N_0) \Rightarrow |y[2m+1]|^2 \sim \text{expon}(N_0)$

$X \sim \text{expon}(\mu)$: $\frac{1}{\mu} e^{-x/\mu}$

Let $Z_1 \triangleq |y[2m+1]|^2$.

$Z_0 \triangleq |y[2m]|^2$.

$Z_0 \sim \text{expon}(a^2+N_0)$, $Z_1 \sim \text{expon}(N_0)$

$f_{Z_0}(z_0) = \frac{1}{a^2+N_0} e^{-z_0/(a^2+N_0)}$; $f_{Z_1}(z_1) = \frac{1}{N_0} e^{-z_1/N_0}$

Z_1 & Z_0 are independent.

$$\Pr(\text{error}) = P\left[Z_1 > Z_0 \,/\, \underline{x} = \underline{x}_A \right] = \int_0^\infty \left[\int_{z_0}^\infty f_{Z_1}(z_1)\, dz_1 \right] f_{Z_0}(z_0)\, dz_0$$

$$= \int_0^\infty \left[\int_{z_0}^\infty \frac{1}{(\cancel{a^2+}N_0)} e^{-z_1/(\cancel{a^2+}N_0)}\, dz_1 \right] \frac{1}{(a^2+N_0)} e^{-z_0/(N_0+a^2)}\, dz_0$$

$$= \frac{1}{N_0(a^2+N_0)} \int_0^\infty \left[\frac{e^{-z_1/(\cancel{a^2+}N_0)}}{(-1/(\cancel{a^2+}N_0))} \Bigg|_{z_0}^{\infty} \right] e^{-z_0/(N_0+a^2)}\, dz_0$$

$$= \frac{1}{a^2+N_0} \int_0^\infty e^{-z_0/(\cancel{a^2+}N_0)}\, e^{-z_0/(N_0+a^2)}\, dz_0$$

$$= \frac{1}{a^2+N_0} \; \frac{e^{-z_0\left[\frac{1}{N_0} + \frac{1}{a^2+N_0}\right]}}{-\left[\frac{1}{N_0} + \frac{1}{a^2+N_0}\right]} \Bigg|_0^{\infty}$$

$$= \frac{1}{1 + \frac{N_0+a^2}{\cancel{a^2+}N_0}} \;\cancel{=}\; \cancel{\frac{a^2+N_0}{a^2+2N_0}} \;\cancel{=}\; \cancel{\frac{1+\frac{a^2}{N_0}}{2+\frac{a^2}{N_0}}}$$

$$= \underline{\underline{\frac{1}{2+\frac{a^2}{N_0}}}}$$

$$* \quad \Pr(\text{error}) = \frac{1}{2 + \frac{a^2}{N_0}}.$$

$\boxed{\text{Lecture 9:}}$ (19 Aug 2008)

Function of average SNR defined as follows.

$$\text{SNR} = \frac{\text{average received signal energy per complex symbol time}}{\text{av. noise energy per complex symbol time}}$$

$$= \frac{(a^2/2)}{N_0} = \frac{a^2}{2N_0}$$

$$\boxed{P_e = \frac{1}{2(1+\text{SNR})}}$$

($P_e = 10^{-3}$ requires SNR ≈ 500 (27dB)
BPSK over AWGN, $P_e = 10^{-3}$ requires ≈ 7dB)

Case 2: $h[m]$ is known at the receiver. (Coherent comm.)

(Needs to be estimated in practice. If coherence time >> symbol time, channel can be estimated with limited training overhead).

BPSK: What is the optimal detection rule in this case?

$$y[m] = h[m]\,x[m] + w[m].$$

(Drop index m for convenience)

$$y = hx + w$$

$w \sim CN(0, N_0)$

h known at receiver.

$x = \pm a$ with equal probability.

* ML is optimal since $\pm a$ are equally likely.

If $x = +a$, $y \sim CN(ha, N_0)$

If $x = -a$, $y \sim CN(-ha, N_0)$.

$$\text{If } |y-ha|^2 > |y+ha|^2, \quad \hat{x} = -a$$

$$|y-ha|^2 < |y+ha|^2, \quad \hat{x} = +a.$$

$$\text{If } |y^2| + |h|^2a^2 - 2\,\mathrm{Re}(h^*y)\,a < |y|^2 + |h|^2a^2 + 2\,\mathrm{Re}(h^*y)\,a$$

$$\hat{x} = a.$$

$$\left[\,(y-ha)(y-ha)^* = |y|^2 + |h|^2a^2 - hay^* - yh^*a\,\right]$$

(Prof: Exercise 3) $\rightarrow$ If $\mathrm{Re}(h^*y) > 0$, $\hat{x} = a$ (assuming $a>0$)

else, $\hat{x} = -a$.

If $x=+a$, $\mathrm{Re}(h^*y) \sim N\left(|h|^2a, \frac{N_0|h|^2}{2}\right)$

If $x=-a$, $\mathrm{Re}(h^*y) \sim N\left(-|h|^2a, \frac{N_0}{2}|h|^2\right)$.

$$\underset{\text{for given } h}{\Pr(\text{error})} = \underset{\text{for a given } h}{\Pr(\text{error} \mid x=+a)} = Q\left(\frac{2|h|^2a}{2\sqrt{\frac{N_0}{2}|h|^2}}\right) = Q\left(\frac{a|h|}{\sqrt{N_0/2}}\right)$$

$$= Q\left(\sqrt{2|h|^2 SNR}\right) \quad \text{where } SNR = \frac{a^2}{N_0}$$

as defined earlier.

This is probability of error for a given h.

P_e = Average probability of error (averaged over h)

$$= E\left[Q\left(\sqrt{2|h|^2 SNR}\right)\right]$$

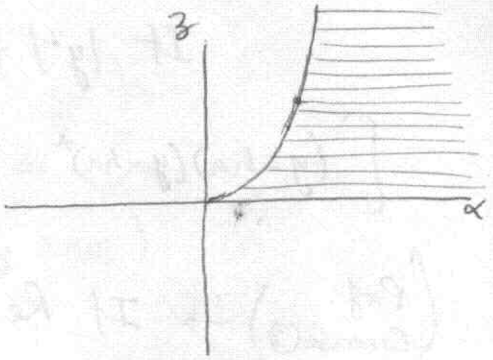
$h \sim CN(0,1) \Rightarrow |h|^2 \sim \text{expon}(1) \quad$ Let $Z \triangleq |h|^2$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\alpha^2/2}\, d\alpha$$

$$Q\left(\sqrt{2|h|^2 SNR}\right) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2|h|^2 SNR}}^\infty e^{-\alpha^2/2}\, d\alpha$$

$\alpha = \sqrt{2z\,SNR}$

$z = \dfrac{\alpha^2}{2\,SNR}$



$$P_e = \int_0^\infty \left[\frac{1}{\sqrt{2\pi}} \int_{\sqrt{2z\,SNR}}^\infty e^{-\alpha^2/2}\, d\alpha\right] e^{-z}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_{\sqrt{2z\,SNR}}^\infty e^{-\alpha^2/2}\, d\alpha\, e^{-z}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^{\alpha^2/2SNR} e^{-\alpha^2/2} e^{-z}\, dz\, d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\alpha^2/2}\left[1 - e^{-\alpha^2/2SNR}\right] d\alpha \quad = \frac{1}{2}\left(1 - \sqrt{\frac{SNR}{1+SNR}}\right)$$

$$= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{\alpha^2}{2}\left(1+\frac{1}{SNR}\right)} d\alpha = \frac{1}{2} - \sqrt{\frac{SNR}{1+SNR}} \underbrace{\frac{1}{\sqrt{2\pi}\sqrt{\frac{SNR}{1+SNR}}} \int_0^\infty e^{-\frac{\alpha^2}{2}\frac{1+SNR}{SNR}} d\alpha}_{1/2}$$

At high SNR

$$\sqrt{\frac{SNR}{1+SNR}} \approx 1 - \frac{1}{2SNR}$$

$$\Rightarrow P_e \approx \frac{1}{4SNR}$$

$P_e = 10^{-3}$, when $SNR \approx 24\,dB$

$$\sqrt{\frac{SNR}{1+SNR}} = \sqrt{\frac{1}{1+\frac{1}{SNR}}} = \left(1+\frac{1}{SNR}\right)^{-\frac{1}{2}}$$

Large SNR $\Rightarrow$ $\frac{1}{SNR}$ is small.

$$f(x) = \frac{1}{\sqrt{1+x}}$$

$$f'(x) = \frac{-1}{2\sqrt{1+x}\,(1+x)}$$

$\boxed{\text{Lecture 10 :}}$ (20/8/2008)

Remarks:

* At high SNR, $Pr(\text{error}) \propto \frac{1}{SNR}$ for the Rayleigh fading channel.

  (We derived exact $Pr(\text{error})$ for binary orthogonal signaling & binary antipodal signaling for the non-coherent & coherent cases respectively).

* Performance is much poorer than the performance on an AWGN channel. For the AWGN channel case, $Pr(\text{error})$ decays exponentially with SNR.

Closer look at why performance is poor:

– What is the main (dominant) error event at high SNR?

For a given h, Prob. of error is $Q\left(\sqrt{2|h|^2 SNR}\right)$.

Even if the average SNR is high,

$|h|^2 SNR$ can still be small if $|h|^2$ is close to zero.

This event is a "deep fade".

$$Pr(\text{deep fade}) = Pr\left(|h|^2 SNR < 1\right)$$

$$= \int_0^{1/SNR} e^{-x}\, dx$$

$$= 1 - e^{-1/SNR}$$

$$\approx \frac{1}{SNR} \quad \text{at high SNR.}$$

Even at high average SNR, $Pr(\text{deep fade})$ is significant and becomes dominant.

For an AWGN channel, dominant error event is noise being large.

<u>Note</u>:

How much does the $\propto \frac{1}{SNR}$ conclusion depend on the Rayleigh fading assumption?

- Important quantity is $Pr\left(|h|^2 < \epsilon\right)$ for small ϵ. ("Deep fade" event)

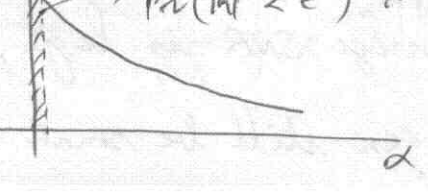
- If PDF of $|h|^2$ is non-zero at 0 and continuous, this is $\approx f_{|h|^2}(0)\,\epsilon \ (\propto \epsilon)$.

  If $Pr(|h|^2 < \epsilon) \propto \epsilon$, $Pr(\text{error}) \propto \frac{1}{SNR}$.

  Therefore, it is also true for other fading models that satisfy the above condition on PDF of $|h|^2$.

Rayleigh model

$|h|^2 \sim$ exponential



$$Pr(|h|^2 < \epsilon) \approx f_{|h|^2}(0)\,\epsilon \Rightarrow Pr\left(|h|^2 SNR < 1\right) \approx \frac{1}{SNR} f_{|h|^2}(0)$$

Performance of QPSK: (Increasing the spectral efficiency)

- In BPSK, we used real symbols. However, we can use complex symbols without increasing bandwidth.
  Two "degrees of freedom" are available. BPSK uses only one. QPSK can use both and still achieve same performance.

$$\text{BPSK} \quad \pm a \qquad SNR_b = \frac{a^2}{N_0}.$$

$$\text{QPSK} \quad (\pm a, \pm a) \qquad SNR_q = \frac{2a^2}{N_0}. \quad \text{(Same energy per bit)}$$

$$\text{Bit error probability in BPSK} = Q\left(\sqrt{2SNR_b}\right) \quad \text{AWGN}$$

$$Q\left(\sqrt{2|h|^2 SNR_b}\right) \quad \text{Fading given } h.$$

Bit error probability in QPSK is the same as in BPSK

(SNR is however twice; $SNR_q = 2SNR_b$)

$$= Q\left(\sqrt{SNR_q |h|^2}\right)$$

$$\text{Average BER} = \frac{1}{2}\left(1 - \sqrt{\frac{SNR_q}{2+SNR_q}}\right) \approx \frac{1}{2SNR_q}$$

(Replace SNR in BPSK result by $\frac{SNR_q}{2}$).

* Good schemes use all the available "$\boxed{\text{degrees of freedom}}$".

If we want to use only real symbols (one degree of freedom) we would have to use 4-PAM to send the same rate. This will be worse in terms of BER.
It is more efficient to pack, for a given dmin, in higher-dimensional space than a lower dimensional space.

→ Differential BPSK / QPSK (exercise) 4

Diversity:

- For the schemes above, $P_e \propto \frac{1}{SNR}$ for high SNR.

  Significant prob. of "deep fade" is the reason.

- Diversity: Ensure that information symbols are received through multiple independent channels. Reliable communication is possible as long as one of these channels is good.

- Diversity can be achieved in time → coding & interleaving (time-selective channel)

  frequency → coding over a freq. selective channel.

  space → coding over space if antennas are able to create independent channels.

- Repetition coding: Simple way to achieve diversity. (for easy understanding) More efficient codes that achieve coding gain / better rate will be also be discussed.

- Diversity in coherent / non-coherent scenario will also be discussed.