EE5040: Adaptive Signal Processing

Problem Set 1: Optimal least-mean-squares estimation

1. (Sayed I.13) Consider noisy observations \( Y_i = X + V_i \), where \( X \) and \( V_i \) are independent real-valued random variables, \( V_i \) is a white-noise Gaussian random process with zero mean and variance \( \sigma_v^2 \), and \( X \) takes the values \( \pm 1 \) with equal probability. The value of \( X \) is the same for all measurements \( \{ Y_i \} \).

   (a) Show that the least-mean-squares estimate of \( X \) in terms of \( \{ Y_0, Y_1, \ldots, Y_{N-1} \} \) is

   \[ \hat{X}_N = \tanh \left( \frac{1}{N-1} \sum_{i=0}^{N-1} \frac{Y_i}{\sigma_v^2} \right). \]

   (b) Assume \( X \) takes the value 1 with probability \( p \) and the value -1 with probability \( 1-p \). Show that the least-mean-squares estimate of \( X \) in terms of \( \{ Y_0, Y_1, \ldots, Y_{N-1} \} \) is

   \[ \hat{X}_N = \tanh \left( \frac{1}{2} \ln \left( \frac{p}{1-p} \right) + \frac{1}{N-1} \sum_{i=0}^{N-1} \frac{Y_i}{\sigma_v^2} \right). \]

   (c) Assume that the noise is correlated. Let \( R_v = \mathbb{E}[VV^T] \), where \( V = [V_0, V_1, \ldots, V_{N-1}]^T \).

   Show that the least-mean-squares estimate of \( X \) in terms of \( \{ Y_0, Y_1, \ldots, Y_{N-1} \} \) is

   \[ \hat{X}_N = \tanh \left( \frac{1}{2} \ln \left( \frac{p}{1-p} \right) + 1^T R_v^{-1} Y \right), \]

   where \( 1 = [1, 1, \ldots, 1]_{N \times 1} \).

2. (Sayed I.16) Suppose we observe \( Y = X + V \), where \( X \) and \( V \) are independent real-valued random variables with exponential distributions with parameters \( \lambda_1 \) and \( \lambda_2 \) (\( \lambda_1 \neq \lambda_2 \)). That is, the PDFs of \( X \) and \( V \) are \( f_X(x) = \lambda_1 e^{-\lambda_1 x} \) for \( x \geq 0 \) and \( f_V(v) = \lambda_2 e^{-\lambda_2 v} \) for \( v \geq 0 \), respectively.

   (a) Using the fact that the PDF of the sum of two independent random variables is the convolution of the individual PDFs, show that

   \[ f_Y(y) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_2 y} \left[ e^{(\lambda_2 - \lambda_1) y} - 1 \right], \quad y \geq 0. \]

   (b) Establish that \( f_{X,Y}(x,y) = \lambda_1 \lambda_2 e^{(\lambda_2 - \lambda_1) x - \lambda_2 y} \), for \( x \geq 0 \) and \( y \geq 0 \).

   (c) Show that the least-mean-squares estimate of \( X \) given \( Y = y \) is

   \[ \hat{X} = \frac{1}{\lambda_1 - \lambda_2} - \frac{e^{-\lambda_1 y}}{e^{-\lambda_2 y} - e^{-\lambda_1 y}}. \]