EE5121: Optimization Methods

HW #2: Topology

1 Sequences

Exercise 1. *Formally prove the following limits.*

1. $\lim_{n\to\infty} \frac{1}{1+n} = 0$.
2. $\lim_{n\to\infty} (-1)^n \exp(-n) = 0$.

Exercise 2. *Prove true or false: (provide a counter example if false and a proof if true)*

1. *Does* $x_n < y_n$ *for all* $n \geq 1$ *imply* $\lim_{n\to\infty} x_n < \lim_{n\to\infty} y_n$*?*
2. *Does* $x_n < y_n$ *for all* $n \geq 1$ *imply* $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$*?*

Exercise 3. *Prove that a monotone increasing and bounded sequence converges to its* sup, i.e., *if* $\{x_n\}$ *is an increasing sequence,*

$$x_n \to \sup(\{x_n\})$$

Exercise 4. *In this problem, we will review two important numbers related to sequences* $\liminf$ *and* $\limsup$ *which are defined as follows.*

$$\liminf x_n = \sup\{\inf\{x_m : \ m > n\} : n > 0\},$$

$$\limsup x_n = \inf\{\sup\{x_m : \ m > n\} : n > 0\}.$$

Let's first consider an example of an increasing sequence $x_n = 1/n, \ n > 1$.

$$\liminf x_n = \sup\{\inf\{\frac{1}{m} : \ m > n\} : n > 0\}.$$

Since $1/m$ *is a decreasing sequence,*

$$\inf\{\frac{1}{m} : \ m > n\} = \inf\{\frac{1}{n+1}, \frac{1}{n+2}, \frac{1}{n+3} \dots\} = 0.$$

Hence $\liminf x_n = \sup\{0, 0, 0, \dots\} = 0$. *We will now compute* $\limsup$

$$\begin{aligned}
\limsup x_n &= \inf\{\sup\{\frac{1}{m} : \ m > n\} : n > 0\} \\
&= \inf\{\sup\{\frac{1}{n+1}, \frac{1}{n+2}, \frac{1}{n+3} \dots\} : n > 0\} \\
&= \inf\{\frac{1}{n+1} : \ n > 0\} = 0.
\end{aligned}$$

Observe that $\liminf$ *and* $\limsup$ *always exist (if we consider extended real numbers* $\mathbb{R} \cup \{\infty, -\infty\}$*), even when the limit does not exist.*

1. *Let* $x_n = (-1)^n$*: Find its* $\limsup$ *and* $\liminf$.

2. *Let* $x_n = 1$

3. *Prove that for any sequence* $\{x_n\}$, $\liminf x_n \leq \limsup x_n$.

4. *If* $\liminf = \limsup$ *show that the limit exists and*

$$\lim x_n = \liminf x_n = \limsup x_n.$$

Hint: Observe that

$$\inf(x_n, x_{n+1}, x_{n+2}, \dots) \leq x_n \leq \sup(x_n, x_{n+1}, x_{n+2}, \dots)$$

and the fact that a monotonic (increasing or decreasing) and bounded sequence converges.

2 Open sets

Exercise 5. *Prove that the intersection of two open sets is open. Also prove that the union of two open sets is open.*

Exercise 6. *This problem emphasizes the fact that the notion of a set being open (or closed) depends on the ambient space in consideration. As a subset of* $\mathbb{R}$ *check if* $(0,1)$ *is an open set. Is* $(0,1)$ *an open set of* $\mathbb{R}^2$*? (Use the definition of open set)?*

Exercise 7. *Let* $A = (0,10) \setminus \{1,2,3,4,5\}$. *Is the set* A *an open subset of* $\mathbb{R}$*?*

Exercise 8. *Let* A *be a* $n \times n$ *invertible matrix and* $b \in \mathbb{R}^n$ *belong to the range space of* A. *Show that the set*

$$\{x : x \in \mathbb{R}^n, Ax < b\},$$

is an open subset of $\mathbb{R}^n$. *What happens if* A *is not invertible?*

Exercise 9. *Observe that the definition of an open set depends on the notion of a Ball which in turn depends on the norm used. Suppose* $A \subset \mathbb{R}^d$ *is an open set with respect to the norm* $\|.\|_2$ *(the standard Euclidean norm). Is the set* A *an open set in* $\mathbb{R}^d$, *when we use a different norm, for example the* $\|.\|_1$ *norm? Hint: Use the equivalence of norms in* $\mathbb{R}^d$ *and the definition of open sets.*

Exercise 10. *Prove that a set* S *is open if and only if all of its elements are inter points,* i.e.$S = S^o$.

Exercise 11. *What is the interior of the set* $x^2 + y^2 \leq 1$ *as a subset of* $\mathbb{R}^3$ *(with respect to the metric on* $\mathbb{R}^3$*). What is its interior as a subset of* $\mathbb{R}^2$ *(with respect to the metric on* $\mathbb{R}^2$*).*

3 Closed sets

Exercise 12. *Check if the following sets are closed subsets of* $\mathbb{R}^2$

$$A_1 = \{(x,y) : x > 0, y > 0, 3x + 2y \leq 1\}.$$

$$A_2 = \{(x,y) : x > 0, y > 0, 3x + 2y > 1\}.$$

$$A_4 = \{(x,y) : 0 < x < \pi, \sin(x) \leq 1/2\}.$$

please provide reasons.

Exercise 13. *Prove that the intersection of two closed sets is closed. Also prove that the union of two closed sets is closed. (Hint: look at their complements and use Exercise 5)*

Exercise 14. *Prove that the supremum and the inifimum of a closed set* $S \subset \mathbb{R}$ *belong to the set. (Hint: Prove by contradiction)*

4 Continuous functions

Exercise 15. *Let* $f(x)$ *and* $g(x)$ *be two continuous functions from* $\mathbb{R}^d$ *to* R*. Also* $g(x) \neq 0$*. Show that*

1. $f(x) + g(x)$ *is a continuous function.*
2. $\frac{f(x)}{g(x)}$ *is a continuous function.*
3. *Is* $\max\{f(x), g(x)\}$ *a continuous function?*

Exercise 16. *In the class we have proved that if a function is continuous, then the inverse image of an open set is open. Now prove the converse: Is the inverse image of any open set is open, then the function is continuous.*

5 Compact sets

Exercise 17. *Are the following sets compact: provide some arguments*

1. $A_1 = [5, 10] \cup \{0\}$
2. $A_2 = \{0, 1/2, 1/3, 1/4, /1/5, ...\}$ *as a subset of* $\mathbb{R}$
3. $A_3 = \{x; x \in \mathbb{R}^d, \|x\|_\alpha \leq 1\}$
4. $A_4 = \{x; x \in \mathbb{R}^d, Ax \geq b\}$*,* A *is invertible* $d \times d$ *matrix and* $b \neq 0$
5. $A_5 = A_4 \cap B(0, 10)$
6. $A_6 = (0, 1)$ *as a subset of* $\mathbb{R}^2$*.*
7. $A_7 = \mathbb{R}$*. Try to prove or disprove using the sequential definition of compactness.*
8. $A_8 = \{x \in \mathbb{R}^{d+1} : x_1 + x_2 + ... + x_{d+1} = 1, x_i \geq 0\}$*, the d-dimensional simplex.*

Exercise 18. *Prove that the union of two compact sets is compact. Similarly prove the intersection.*

Exercise 19. *In the class we have proved that every bounded and closed subset of* $\mathbb{R}^d$ *is compact. Now prove the converse: Every compact subset of* $\mathbb{R}^d$ *is closed and bounded.*