Motion-free superresolution and the role of relative blur

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1. INTRODUCTION

Superresolution (SR) is the process of deriving a high-resolution image from several low-resolution observations approximately covering the same region in a scene. The observations contain similar but not identical information. Typically, SR methods assume that motion exists between the camera and the scene, we would be able to do without the correspondence problem inherent in SR techniques involving subpixel shifts of the camera. Elad and Feuer give a brief analysis of this question and show that SR is possible even without motion. This is also termed motionless or motion-free SR. They demonstrate that one can get a superresolved image from several low-resolution observations computed with different degrees of defocusing. The aim of SR for the motion-free case is to undo the effects of blurring and aliasing by making use of the information in the given set of observations.

There are two parts to this paper. In the first part, we derive analytically the relation for the reconstruction of the superresolved image from its blurred and downsampled versions. No relative motion is assumed. In the classic paper of Papoulis on the generalized sampling expansion, it was shown (in the continuous-time domain) that a band-limited one-dimensional (1-D) function \( f(t) \) is uniquely determined in terms of the samples of the response of \( M \) linear time-variant systems with input \( f(t) \) and sampled at the Nyquist rate. Brown later showed that reconstruction is also possible by passing \( f(t) \) through \( M \) linear time-invariant filters provided that the filters are independent in a certain sense. In this paper, we assume that the original two-dimensional (2-D) image has already been sampled at the Nyquist rate and that only samples \( f[m, n] \) of it are available. Given blurred and downsampled versions of this image, we derive analytical expressions in the discrete Fourier transform (DFT) domain for the reconstruction of \( f[m, n] \) from the given observations. Our analysis leads to the construction of an appropriate system matrix whose inverse yields the desired reconstruction filters for obtaining the superresolved image.

This can be alternatively looked upon as the problem of reconstructing \( f[m, n] \) from samples of the responses of \( M^2 \) linear time-invariant filters to \( f[m, n] \) downsampled by a factor of \( M \). The formulation is quite general and leads to an interesting parallel to the 1-D results in Refs. 2 and 12 but in the DFT domain.

In the second part of the paper, we examine the effect of relative blurring among the observations on the quality of the superresolved image. Although earlier studies have...
shown that blurred and downsampled observations can yield a superresolved image, the effect of relative blurring on the well-posedness of the motion-free SR problem has not been addressed. We derive the Cramer–Rao lower bound (CRLB) on the covariance of the error in the estimate of the superresolved image for an unbiased estimator and show that the CRLB depends on the condition number of the system matrix. Our study reveals that for better conditioning and stability of the SR problem, the blur span should be rich. Merely having a greater number of observations is not enough. What really affects the quality of the superresolved image is the dependence among the blur kernels that yield the measured images. These are also validated through several simulations.

Interestingly, the effect of relative blur has been studied in different contexts elsewhere too. For the problem of depth from defocused images, it is shown in Ref. 13 that a good estimate of depth depends on the relative blurring between two defocused images. Ghiglia\textsuperscript{14} had also made similar observations but in the context of image restoration.

The organization of the paper is as follows. In Section 2, we carry out image reconstruction analysis for the motion-free SR problem. A quantitative study of the effect of relative blurring on the accuracy of the superresolved image in terms of the CRLB is performed in Section 3. Simulation results are discussed in Section 4. The paper concludes with Section 5.

2. ANALYSIS OF MOTION-FREE SUPERRESOLUTION

We assume that the 2-D continuous image function  \( x(t, \tau) \) has been sampled at the Nyquist rate to yield the discrete image  \( x[m, n] \) of size  \( N \times N \). In this section, we derive analytically the relation between the observations and the original superresolved image in the DFT domain. The blurred and downsampled observation sequences are expressed as the product of a suitably derived system matrix with the original superresolved image  \( x[m, n] \).

We denote the discrete-time Fourier transform (DTFT) of the original image  \( x[m, n] \) by  \( X(\exp(\jmath \omega_1), \exp(\jmath \omega_2)) \). Let the frequency spread of  \( x[m, n] \) be from  \(-\omega_N\) to  \( \omega_N\).

Let the point-spread function of the \( k \)th blur kernel be  \( h_k[m, n] \) (also of size  \( N \times N \)) with DTFT  \( H_k(\exp(\jmath \omega_1), \exp(\jmath \omega_2)) \), which vanishes outside  \((-\omega_2, \omega_2)\). The \( k \)th observation  \( Y_k[m, n] \) is obtained by blurring  \( x[m, n] \) with the point-spread function  \( h_k[m, n] \) and then downsampling the result by a factor of  \( M \) along both the row and column directions. For the case of exact reconstruction, downsampling by a factor of  \( M \) implies that we need  \( M^2 \) observations; i.e., the image  \( x[m, n] \) must be passed through  \( M^2 \) different blur kernels  \( h_k[m, n], k = 1, 2, \ldots, M^2 \).

The DTFT of the  \( k \)th observation  \( Y_k[m, n] \) can then be written\textsuperscript{15} as

\[
Y_k(\exp(\jmath \omega_1), \exp(\jmath \omega_2)) = \frac{1}{M^2} \sum_{p=0}^{M-1} \sum_{q=0}^{M-1} X\left(\exp\left(\jmath \frac{\omega_1 - 2\pi p}{M}\right), \exp\left(\jmath \frac{\omega_2 - 2\pi q}{M}\right)\right) \times H_k\left(\exp\left(\jmath \frac{\omega_1 - 2\pi p}{M}\right), \exp\left(\jmath \frac{\omega_2 - 2\pi q}{M}\right)\right). \tag{1}
\]

Equation (1) can be interpreted as follows. The spectra of  \( x[m, n] \) and  \( h_k[m, n] \) are each scaled by  \( M \). Then,  \( M^2 \) versions of their shifted product are added up to generate the output  \( Y_k(\exp(\jmath \omega_1), \exp(\jmath \omega_2)) \). Clearly, if there is no aliasing of the shifted spectra, the use of an appropriate low-pass filter would give us all the information needed to reconstruct  \( x[m, n] \), and one observation would suffice. Of course, this assumes that the blur kernel does not vanish in the range  \((-\omega_N, \omega_N)\). However, when there is aliasing,  \( x[m, n] \) cannot be recovered from a single observation.

For notational convenience, let the 2-tuple

\[
a_{p,q} = \left(\exp\left(\jmath \frac{\omega_1 - 2\pi p}{M}\right), \exp\left(\jmath \frac{\omega_2 - 2\pi q}{M}\right)\right).
\]

Given  \( M^2 \) observations  \( Y_k, k = 1, 2, \ldots, M^2 \), the relation in Eq. (1) can be rearranged in matrix–vector form as

\[
Y = HX,
\]

where

\[
Y = [Y_1(\exp(\jmath \omega_1), \exp(\jmath \omega_2)), Y_2(\exp(\jmath \omega_1), \exp(\jmath \omega_2)), \ldots, Y_l(\exp(\jmath \omega_1), \exp(\jmath \omega_2)), \ldots, Y_{M^2}(\exp(\jmath \omega_1), \exp(\jmath \omega_2))]^T,
\]

\[
X = [X(a_{0,0}), \ldots, X(a_{0,M-1}), X(a_{1,0}), \ldots, X(a_{1,M-1}), \ldots, X(a_{M-1,0}), \ldots, X(a_{M-1,M-1})]^T,
\]

\[
H = \begin{bmatrix}
H_1(a_{0,0}) & \cdots & H_1(a_{0,M-1}) & H_1(a_{1,0}) & \cdots & H_1(a_{1,M-1}) & \cdots & H_1(a_{M-1,0}) & \cdots & H_1(a_{M-1,M-1}) \\
H_2(a_{0,0}) & \cdots & H_2(a_{0,M-1}) & H_2(a_{1,0}) & \cdots & H_2(a_{1,M-1}) & \cdots & H_2(a_{M-1,0}) & \cdots & H_2(a_{M-1,M-1}) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
H_{M^2}(a_{0,0}) & \cdots & H_{M^2}(a_{0,M-1}) & H_{M^2}(a_{1,0}) & \cdots & H_{M^2}(a_{1,M-1}) & \cdots & H_{M^2}(a_{M-1,0}) & \cdots & H_{M^2}(a_{M-1,M-1})
\end{bmatrix}.
\]
Here, T is the transpose operator. Note that each of the terms in the vector \( X \) gives only a part of the entire frequency spectrum. For example, for \( \omega_1 \in (-\pi, \pi) \) and \( \omega_2 \in (-\pi, \pi) \), the term
\[
X \left( \exp \left( j \frac{\omega_1 - 2\pi p}{M} \right), \exp \left( j \frac{\omega_2 - 2\pi q}{M} \right) \right)
\]
generates the frequency spectrum in the ranges
\[
\left( \frac{-\pi - 2\pi p}{M}, \frac{\pi - 2\pi p}{M} \right), \quad \left( \frac{-\pi - 2\pi q}{M}, \frac{\pi - 2\pi q}{M} \right).
\]
Thus each term generates only \( 1/M^2 \) of the entire frequency spectrum of \( x[m, n] \), and the \( M^2 \) terms together yield \( x[m, n] \).

The blur kernels are general but not entirely arbitrary. From Eq. (2), we note that \( X \) can be reconstructed only if the matrix \( H \) is nonsingular. Hence we need \( M^2 \) linearly independent blur kernels. Reconstruction will also not be possible if none of the blur sequences spans any particular frequency range of
\[
X \left( \exp \left( j \frac{\omega_1 - 2\pi p}{M} \right), \exp \left( j \frac{\omega_2 - 2\pi q}{M} \right) \right).
\]
For example, take the case where
\[
\omega_k < \omega_N \quad \forall k \in (1, M^2).
\]
Surely, \( H \) will then turn out to be singular. The above condition translates to loss of frequency content during the low-pass blurring operation. Hence exact reconstruction methods can only reverse the effects of aliasing. If information is lost over any interval within \((-\omega_N, \omega_N)\) during blurring, the original image \( x[m, n] \) cannot be regenerated.

We know that the DFT of a sequence of length \( N \) can be obtained from its DTFT by sampling it at \( N \) points. In the motion-free SR problem, \( x[m, n] \) is an \( N \times N \) image that has been blurred and downsampled by a factor of \( M \) to yield the observation \( y_k[m, n] \) of size \( (N/M)(N/M) \times (N/M) \). Using Eq. (1), we sample the DTFT of the \( k \)th observation \( y_k[m, n] \) at \( N/M \) points to obtain
\[
Y_k(l_1, l_2) = \frac{1}{M^2} \sum_{p=0}^{M-1} \sum_{q=0}^{M-1} X \left( \frac{2\pi l_1}{N} - \frac{2\pi p}{M}, \frac{2\pi l_2}{N} - \frac{2\pi q}{M} \right) \times H_k \left( \frac{2\pi l_1}{N} - \frac{2\pi p}{M}, \frac{2\pi l_2}{N} - \frac{2\pi q}{M} \right),
\]
\[
l_1, l_2 = 0, 1, ..., \frac{N}{M} - 1.
\]
The relation for the DFT of \( y_k[m, n] \) in terms of the DFTs of \( x[m, n] \) and \( h_k[m, n] \) can then be written as
\[
Y_k(l_1, l_2) = \frac{1}{M^2} \sum_{p=0}^{M-1} \sum_{q=0}^{M-1} X \left( \frac{l_1 - pN}{M}, \frac{l_2 - qN}{M} \right) \times H_k \left( \frac{l_1 - pN}{M}, \frac{l_2 - qN}{M} \right),
\]
\[
l_1, l_2 = 0, 1, ..., \frac{N}{M} - 1,
\]
where \( (x - y)_N \) denotes the mod \( N \) value of \( x - y \). Each observation \( Y_k(l_1, l_2) \) yields \( N^2/M^2 \) terms, and if one has \( M^2 \) independent observations, then one can solve for the \( N^2 \) DFT coefficients of \( X(l_1, l_2) \) and reconstruct the original image \( x[m, n] \). Equation (3) can also be written as
\[
Y_k(N^2/M^2) \times 1 = H_k(N^2/M^2) \times N \cdot X(N^2 \times 1), \quad (4)
\]
where

\[
Y_k = \begin{bmatrix} Y_k(0, 0) & \cdots & Y_k(0, N/M - 1) & Y_k(1, 0) & \cdots & Y_k(1, N/M - 1) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\
Y_k(N/M - 1, 0) & \cdots & Y_k(N/M - 1, N/M - 1) \end{bmatrix}^T,
\]
\[
X = \begin{bmatrix} X(0, 0) & X(0, 1) & \cdots & X(0, N/M - 1) & X(1, 0) & \cdots & X(0, N - 1)X(1, 0) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
X(N/M - 1, 0) & \cdots & X(N/M - 1, N - 1) & X(N - 1, 0) & \cdots & X(N - 1, N - 1) \end{bmatrix}^T. \quad (5)
\]
The matrix $H_k$ can be succinctly expressed as

$$
H_k = \begin{bmatrix}
H_k^{T} 0,0 & H_k^{T} 0,1 & \cdots & H_k^{T} (N/M) - 1, 0 & \cdots & H_k^{T} 1, 0 & \cdots & H_k^{T} 1, 1 & \cdots & H_k^{T} (N/M) - 1, 1 & \cdots & H_k^{T} (N/M) - 1, (N/M) - 1 \end{bmatrix},
$$

where the row vector $H_k i_1, i_2$ can be shown to be

$$
H_k i_1, i_2 = \begin{bmatrix}
0 & 0 & \cdots & 0 & H_k i_1, i_2 & 0 & \cdots & 0 & H_k i_1, i_2 + \frac{N}{M} & 0 & \cdots & 0 & H_k i_1, i_2 + \frac{2N}{M} & 0 & \cdots & 0 & H_k i_1, i_2 + \frac{(M-1)N}{M} & 0 & \cdots & 0
\end{bmatrix},
$$

for any of the $M^2$ observations and for any $i_1, i_2 = 0, 1, \ldots, (N/M) - 1$, as that would render one or more rows of the system matrix to be all zeros. The inverse of the system matrix given in Eqs. (10), in effect, yields the reconstruction filters (in the DFT domain) for obtaining the superresolved image from the observations. The similarity of these results to the continuous-time case in Refs. 2 and 12 is particularly noteworthy.

It must be mentioned here that, apart from the invertibility of $H_k$, a pertinent issue to address is the effect of relative blurring among the kernels on the accuracy of the estimate of $X$. This is because as the blur kernels tend to become linearly dependent, even if $H$ is invertible, one would expect the quality of the reconstructed image to degrade, since the observations then carry more or less similar information. A quantification of this effect is important and is discussed next. It is also interesting to analyze the case when there are more than $M^2$ observations available. If there are $L$ observations, where $L > M^2$, then a least-squares (LS) estimate of $X$ can be obtained from the equation

$$
(H^T H)_{LS} = H^T Y,
$$

where

$$
Y = [Y_1^T \quad Y_2^T \quad \cdots \quad Y_L^T]^T, \quad H = [H_1^T \quad H_2^T \quad \cdots \quad H_L^T]^T.
$$

One would expect that an increase in the number of observations provides better results. It will be shown in the following sections that the mere presence of an extra observation is not enough for improvement in the quality of reconstruction. Rather, what really matters is how different the observations are with respect to one another.
3. CRAMÉR–RAO LOWER BOUND AND CONDITION NUMBER ANALYSIS

The problem of SR is ill posed at its worst and ill conditioned at its best. Smoothness is a common constraint that is often used for overcoming ill-posedness. For the motion-free SR problem, even if $H$ is invertible, in the presence of noise the ill-conditionedness of $H$ will affect the quality of the result. We quantitatively analyze the effect of relative blur on the accuracy of the estimate of the reconstructed superresolved image. The analysis is based on the Cramér–Rao lower bound (CRLB), which provides a fundamental limit on the variance of the error attainable with an estimator for an unknown parameter. The CRLB expresses the minimum error variance of any estimator $\hat{x}(y)$ of $x$ in terms of the conditional density $f(y|x)$ of the data. In this section, we discuss the application of this well-known mathematical theory to the motion-free SR problem. The CRLB depends on the system matrix, and the study is carried out for both the exact and LS cases.

A. Exact Case

For exact reconstruction, we need $M^2$ observations, where the $i$th measurement is given by

$$Y_i = H_iX + \eta_i, \quad i = 1, \ldots, M^2.$$  \hfill (12)

Here, $H_i$ is of the form given in Eq. (7). The vector $\eta_i$ represents zero-mean white Gaussian noise with variance $\sigma^2$, and $\eta_i$ is assumed to be statistically independent of $\eta_j$ for $i \neq j$. By stacking up the observations in Eq. (12), we get

$$Y = HX + \eta,$$  \hfill (13)

where

$$Y = [Y_1^T \ Y_2^T \ \ldots \ Y_{M^2-1}^T \ Y_{M^2}^T]^T,$$

$$\eta = [\eta_1^T \ \eta_2^T \ \ldots \ \eta_{M^2-1}^T \ \eta_{M^2}^T]^T,$$

$$H = [H_1^T \ H_2^T \ \ldots \ H_{M^2-1}^T \ H_{M^2}^T]^T.$$  \hfill (14)

If $X$ is of dimension $N^2 \times 1$, then each of the $Y_i$’s has dimension $(N^2/M^2) \times 1$. Hence $H_i$ is of size $(N^2/M^2) \times N^2$, while $H$ is a square matrix of size $N^2 \times N^2$. Each of the $H_i$’s is assumed to be of full rank, i.e., rank ($H_i$) $= N^2/M^2$. If the blur kernels are linearly independent of one another, then $H$ will be full rank and invertible.

For an unbiased estimator, the CRLB (Ref. 16) on the covariance of the estimate of $X$ is given by

$$\mathcal{E}[(X - \hat{X})(X - \hat{X})^T] \geq J^{-1},$$

where $\mathcal{E}$ is the expectation operator, $\hat{X}$ is an unbiased estimator of $X$, and

$$J = -\mathcal{E} \left[ \frac{\partial^2}{\partial X^2} \log f(Y) \right].$$

Here, $f(Y)$ is the probability density function of $Y$ given $X$. Since the $\eta_i$’s are assumed to be Gaussian and independent with variance $\sigma^2$, we have

$$f(Y) = \frac{1}{(2\pi\sigma^2)^{N^2/2}} \exp \left[ -\frac{(Y - HX)^T(Y - HX)}{2\sigma^2} \right].$$

Thus

$$\frac{\partial}{\partial X} \log f(Y) = -\frac{1}{2\sigma^2} (2H^TY + 2H^THX),$$

$$\frac{\partial^2}{\partial X^2} \log f(Y) = -\frac{1}{\sigma^2} H^TH,$$

Hence

$$J = -\mathcal{E} \left[ \frac{\partial^2}{\partial X^2} \log f(Y) \right] = \frac{1}{\sigma^2} H^TH,$$

and the CRLB for the exact case turns out to be

$$\mathcal{E}[(X - \hat{X})(X - \hat{X})^T]_{\text{exact}} = \sigma^2 (H^TH)^{-1}. \quad (15)$$

Here, $H^TH = \sum_{i=1}^{M^2} H_i^TH_i$. Although the $H_i$’s themselves are not symmetric, the matrix $H_i^TH_i$ is. Moreover, it is positive semidefinite, since it is obtained as the product of the transpose of a matrix with itself. If the blur kernels are linearly independent, then $H^TH$ is positive definite and hence invertible.

From Eq. (13), an estimate of $X$ can be found as

$$(\hat{X}) = H^{-1}Y = X + H^{-1}\eta,$$  \hfill (16)

where $\hat{X}$ is random and $\mathcal{E}[\hat{X}] = \mathcal{E}[X + H^{-1}\eta] = X$. Also,

$$\mathcal{E}[(X - \hat{X})(X - \hat{X})^T] = \mathcal{E}[(H^{-1}\eta)(H^{-1}\eta)^T] = \sigma^2 (H^TH)^{-1}.$$  

Thus $\hat{X}$ as given by Eq. (16) is an unbiased and efficient estimator, as it meets the CRLB.

From relation (15), we note that for a given noise variance, the CRLB depends on the condition number of $H^TH$. If this matrix is ill conditioned, then the error in the estimate of $X$ is likely to be large, since the CRLB itself will be large. The condition number measures the sensitivity or the vulnerability of a solution. The condition number of a matrix $S$ is given by $\text{cond}(S) = ||S|| \cdot ||S^{-1}||$, where $|| \cdot ||$ denotes the spectral norm. Hence

$$\text{cond}(S) = \left[ \frac{\lambda_{\text{max}}(S^TS)}{\lambda_{\text{min}}(S^TS)} \right]^{1/2}.$$  \hfill (17)

Since the matrix $H^TH$ is symmetric,

$$\text{cond}(H^TH) = \frac{\lambda_{\text{max}}(H^TH)}{\lambda_{\text{min}}(H^TH)},$$

where $H$ as given in Eq. (14).

The effect of $\text{cond}(H^TH)$ as one of the blur matrices (say $H_{M^2}$) becomes linearly dependent on the other blur matrices is studied next. For this purpose, construct matrices

$$a = [H_1^T \ H_2^T \ \ldots \ H_{M^2-1}^T \ H_{M^2}^T]^T,$$

$$b = [H_1^T \ H_2^T \ \ldots \ H_{M^2-1}^T \ H_{M^2}^T]^T,$$
where $H_{M^2}$ is a linear combination of the blur kernels $H_1, H_2, \ldots, H_{M^2-1}$.

If all the $H_i$'s are independent, then $1 \leq \text{cond}(a^T a) < \infty$. Since the matrix $A$ is the same as $H$ in Eq. (14), 
$\text{cond}(H^T H) = \text{cond}(a^T a)$. We wish to study what happens to $\text{cond}(a^T a)$ as $a$ comes closer to $b$.

Let $A = a^T a$ and $B = b^T b$.

Therefore
\[
\text{cond}(a^T a) = \text{cond}(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)},
\]
\[
\text{cond}(b^T b) = \text{cond}(B) = \frac{\lambda_{\max}(B)}{\lambda_{\min}(B)}.
\]

Since $B = b^T b$ is not full rank, $\lambda_{\min}(B) = 0$. Hence $\text{cond}(B) = \infty$.

As $H_{M^2}$ tends to $H_{M^2}$, the eigenvalues of $A$ and $B$ become more and more similar. Clearly, when $A$ becomes equal to $B$, their eigenvalues will be identical, but the SR problem becomes intractable. Hence it is of interest to examine the manner in which the eigenvalues of $A$ and $B$ are related as $A$ tends to $B$. This is given by the classical Weyl's perturbation theorem, which yields bounds on the worst-case absolute error between the exact and perturbed eigenvalues of Hermitian matrices in terms of the two-norm of their difference matrix.

**Theorem 1: Weyl's Perturbation Theorem.** Let $A$ and $B$ be Hermitian matrices with eigenvalues $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ and $\lambda_1(B) \geq \cdots \geq \lambda_n(B)$, respectively. Then
\[
\max_{1 \leq j \leq n} |\lambda_j(A) - \lambda_j(B)| = \|A - B\|.
\]

Here, $\lambda_j$ indicates arrangement of the eigenvalues in decreasing order, and $\|\cdot\|$ denotes the spectral norm. In fact, $\lambda_i(A)$, $1 \leq j \leq n$, are continuous functions on the space of Hermitian matrices. It is customary to state this result as a perturbation theorem, whereby $A$ is a perturbation of $B$; that is, $A = B + C$. The objective is to give bounds for the distance of $\lambda(A)$ from $\lambda(B)$ in terms of $C = A - B$.

By the definition of the spectral norm, relation (17) can be equivalently written as
\[
\max_{j} |\lambda_j(A) - \lambda_j(B)| \leq \lambda_{\max}(A - B).
\]

From relations (17) and (18), it follows that the bounds for the distances of the maximum and minimum eigenvalues of $A$ from those of $B$ can be written as
\[
|\lambda_{\max}(A) - \lambda_{\max}(B)| \leq \lambda_{\max}(A - B),
\]
\[
|\lambda_{\min}(A) - \lambda_{\min}(B)| \leq \lambda_{\max}(A - B).
\]

Now $A = a^T a$ is positive definite ($\lambda_j > 0$), while $B = b^T b$ is positive semidefinite ($\lambda_j \geq 0$), since $B$ is rank deficient. As $A$ approaches $B$, $\lambda_{\max}(A - B)$ decreases. Hence the upper bound on the absolute value of the difference in the minimum and maximum eigenvalues of $A$ and $B$ decreases. Because $\lambda_{\min}(B) = 0$ in relation (19), Weyl's perturbation theorem tells us that the condition number of $A$ will worsen. In the limit when $H_{M^2} = H_{M^2}$, cond $(A)$ becomes equal to cond $(B) = \infty$ and $H^T H$ is no longer invertible, as expected.

**B. Least-Squares Case**

Assume that we have an additional batch of observations, i.e., $Y_i$, $i = 1, \ldots, 2M^2$. Then
\[
Y = HX + \eta,
\]
where
\[
H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \quad H_1 = [H_1^T \cdots H_{M^2}^T]^T,
\]
\[
H_2 = [H_{M^2+1}^T \cdots H_{2M^2}^T]^T.
\]

Here, $Y = [Y_1^T \cdots Y_{M^2}^T Y_{M^2+1}^T \cdots Y_{2M^2}^T]^T$ and $\eta = [\eta_1^T \cdots \eta_{M^2}^T \eta_{M^2+1}^T \cdots \eta_{2M^2}^T]^T$. Following a similar procedure as that discussed in Subsection 3.A for the CRLB analysis of the exact case, it can be shown that
\[
\tilde{C}[(X - \bar{X})(X - \bar{X})^T]_{LS} = \sigma_\eta^2 (H^T H)^{-1},
\]
where $H$ is as given in Eqs. (21). Thus the CRLB for the least-squares (LS) case depends on the condition number of the matrix $H^T H = \bar{H}_1^T \bar{H}_1 + \bar{H}_2^T \bar{H}_2$, where $\bar{H}_1$ and $\bar{H}_2$ are as given in Eqs. (21).

Let $A = \bar{H}_1^T \bar{H}_1$ and $E = \bar{H}_2^T \bar{H}_2$ and both $\bar{H}_1$ and $\bar{H}_2$ be full rank. Therefore $A$ and $E$ are both positive definite. Hence
\[
\text{CRLB}_{LS} = \sigma_\eta^2 (A + E)^{-1}.
\]

We now compare the CRLB for the LS case with that for the exact case, for which we had
\[
\text{CRLB}_{ex} = \sigma_\eta^2 A^{-1}.
\]

Note that $A$ is the same for both the exact and LS situations. Since $A$ and $E$ are positive definite and invertible, $A \equiv (A + E)$ and hence $A^{-1} \equiv (A + E)^{-1}$. Therefore
\[
\text{CRLB}_{LS} \equiv \text{CRLB}_{ex},
\]
which means that the covariance of the error in the estimate of $X$ improves with additional observations.
It must be noted, however, from relation (22) that the CRLB for the LS case depends on the condition number of $A + E$, i.e., $H^T H$. If this matrix is ill conditioned, then, despite more observations, the error in the estimate of $X$ can still be large. The condition number for the LS case is given by

$$\text{cond}(H^T H) = \text{cond}(A + E) = \frac{\lambda_{\text{max}}(A + E)}{\lambda_{\text{min}}(A + E)},$$

(23)

where $H$ is as given in Eqs. (21). The mere presence of $H_2$ in Eqs. (21) does not automatically imply that the reconstruction will be good. Any improvement in accuracy will actually depend on how different $H_2$ is from $H_1$ or, equivalently, how different $A$ is from $E$. In the limit when $E = A$, $\text{cond}(A + E)$ becomes equal to $\text{cond}(A)$. Since the covariance matrix of the error in the estimate of $X$ is lower bounded by the inverse of $H^T H$, this shows that adding more observations without enriching the blur

Fig. 2. Different measurements and superresolved output of the “insect” image corresponding to (a) case 1, (b) case 2, and (c) case 3. In each column, the first five are measured images. The sixth and seventh images correspond to reconstruction using the exact and least-squares (LS) methods, respectively.
Table 1. Condition Number for Each of the Cases Considered in the Simulations

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>Least-Squares</td>
<td>Exact</td>
</tr>
<tr>
<td>$1.23 \times 10^{11}$</td>
<td>$1.12 \times 10^{11}$</td>
<td>$2.43 \times 10^{10}$</td>
</tr>
</tbody>
</table>

Fig. 3. Different observations and superresolved image of “Lena” for (a) case 1, (b) case 2, and (c) case 3. The first five images in each column are the measurements. The sixth and seventh images correspond to reconstruction results using the exact and LS methods, respectively.
span does not improve the stability of the estimation problem at hand.

Yet another interesting fact is that if the additional observation is too severely blurred, then the entries of $E$ in Eq. (23) will be very small and the condition number remains virtually unchanged. This is to be expected because a severely blurred observation is almost homogeneous in nature and contains no new information to improve the reconstruction process.

4. SIMULATION RESULTS
In this section, we study the quality of the reconstructed superresolved image as a function of blur. In the simu-
lations, the blur kernels are chosen to be Gaussian, while the downsampling factor used is 2. The idea is to show how relative blurring affects the output result.

We have considered both exact and LS reconstruction. Three different original images of size $128 \times 128$ pixels were used, and these are shown in Fig. 1. These images are blurred by a Gaussian kernel and downsampling to size $64 \times 64$ pixels. White Gaussian noise of standard deviation 1.0 was added to these images to yield observations of size $64 \times 64$ pixels. The blur parameter for the $i$th Gaussian kernel is denoted by $\sigma_i$. Different values of $\sigma_i$ yield different blur kernels and hence different observations (see Fig. 2). Since the downsampling factor is 2, four observations are required for exact reconstruction, and these are generated by using four different values of $\sigma_i$. A fifth measurement was also considered to demonstrate the performance of the LS method. Simulation results are given for three different sets of blur parameters as follows:

Case 1: $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\sigma_3 = 0.6$, $\sigma_4 = 0.8$, $\sigma_5 = 0.9$;

Case 2: $\sigma_1 = 0.2$, $\sigma_2 = 0.5$, $\sigma_3 = 0.8$, $\sigma_4 = 1.1$, $\sigma_5 = 1.4$;

Case 3: $\sigma_1 = 0.2$, $\sigma_2 = 0.7$, $\sigma_3 = 1.2$, $\sigma_4 = 1.7$, $\sigma_5 = 2.0$.

The values of $\sigma_i$ are chosen as above on purpose to demonstrate the effect of relative blurring on quality of reconstruction and to show the amount of improvement in the quality of the output image due to additional measurements.

In the first experiment, we consider the “insect” image shown in Fig. 2. Five measured images of size $64 \times 64$ pixels obtained by using the $\sigma_i$'s corresponding to case 1 are given in Fig. 2(a). Note that the observations are blurred to different extents depending on the value of $\sigma_i$. Because the blur parameters are known, the matrix $H$ is computed in the DFT domain from Eqs. (7) and (8) by using the $\sigma_i$'s corresponding to case 1. Thus $H_1$, $H_2$, $H_3$, $H_4$, and $H_5$ are obtained. From these matrices, one can obtain the system matrix $H$ given by Eqs. (10). Similarly, the DFT of each of the observations is taken according to Eq. (5), and these are then stacked lexicographically as in Eqs. (10) to get $Y$. To solve for the superresolved image, we must invert the system matrix $H$ in Eq. (9). However, as the original image dimensions are large ($128 \times 128$ pixels), it would be computationally very complex to directly invert $H$. Instead, we use the well-known algebraic reconstruction method by projections due to Kaczmarz (given in Ref. 19) to solve for $X$. The matrix $H$ is very sparse and is treated as such for ease of implementation. The inverse DFT of $X$ thus computed yields the desired result. The superresolved images for case 1 are shown for both the exact and LS methods as the sixth and seventh images in Fig. 2(a), respectively. Note that the quality of the reconstructed image is poor and quite patchy. From the result for the LS case, it is interesting to note that even the additional measurement (i.e., the fifth observation) did not yield much of an improvement over the exact case. This is because the blur parameter $\sigma_5$ for case 1 is quite close to $\sigma_4$. Hence the effective blur span is not sufficiently enhanced. This result corroborates the discussions in Subsection 3.B on the LS estimator.

Next, the two other sets of values of $\sigma_i$ were used, namely those corresponding to cases 2 and 3 on the same insect image. The measured images and the corresponding output for these cases are given in Figs. 2(b) and 2(c) for exact as well as LS, reconstruction. Note that the quality of the superresolved image is different now. Although the matrix $H$ in Eq. (9) is invertible in each of the cases, the quality of the reconstructed image is affected by the degree of relative blur. As the relative blur increases, the condition number of $H$ improves. This can also be seen from Table 1, where the condition number is given for each of the cases considered in the simulations. As the observations come closer, the ill-conditionedness increases and results in a poor-quality output. This is in accordance with our earlier analysis based on Weyl’s perturbation theorem. For case 2, we note that the result corresponding to the exact case is better than that for case 1. This is because the blur span for case 2 is larger than that for case 1. The addition of a fifth measurement improves the quality considerably as the relative blur further increases. A comparison of all the outputs in Fig. 2 reveals that the result corresponding to case 3 is the best. This is also reflected in the condition number being lowest for case 3 in Table 1. Even exact reconstruction for this case is quite good. This is because among all three cases, the relative blur for case 3 is the highest, as $\sigma_1 = 0.2$ and $\sigma_4 = 1.7$. When we add a new observation and derive the LS output, we note that the result further improves. The above experiments were then repeated for the two other images also, namely the “Lena” and “tree” images, and the corresponding results are shown in Figs. 3 and 4, respectively. Results are again given for both exact and LS situations. The set of values of $\sigma_i$ was the same as that used for the insect image. The figures are quite self-explanatory. Inferences similar to the ones drawn in the previous experiment regarding performance of exact and LS methods for cases 1–3 hold well here too. The quality of the reconstructed image and the stability of the problem improve as the relative blur increases. By comparing the reconstructed images with the original images in Fig. 1, we again find that the results are the best for case 3, as expected.

### 5. CONCLUSIONS

In this paper, we first examined the theory of reconstructing a superresolved image from its blurred and downsampled versions. Analytical expressions were derived in the discrete Fourier transform domain that enabled us to express the reconstruction filters as the inverse of a suitably constructed system matrix. Based on the Cramér–Rao lower bound and analysis of the condition number of the system matrix, the effect of relative blurring on the accuracy of the estimated superresolved image was studied. As the observations became more and more similar, the quality of the reconstructed image deteriorated. Using the least-squares estimator, we also showed that the mere presence of a greater number of ob-
servations does not necessarily translate into good reconstruction capability. In fact, it is the linear independence of the blur kernels across the measured images that affects the quality of the result. The study shows that a proper choice of blur kernels is necessary for good-quality reconstruction.

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