EE6001: Stochastic Modeling and the theory of Queues

Tutorial 3

Due on 3/2/2012 by 4pm

1. (i) Prove that convergence in probability implies convergence in distribution

   (ii) Give a counter-example to show that the converse to part (i) does not hold in general.

   (iii) If a sequence of random variables converge in distribution to a real constant C, show that the sequence also converges in probability to C. Thus, if the limit is a constant, convergence in probability and convergence in distribution are equivalent. This is a partial converse to part (i). *Hint:* there is no loss of generality in taking $C = 0$.

2. Exercise 1.45, the one about mosquitoes.

3. The purpose of this exercise is to prove and understand a fundamental result in probability theory, known as the Borel-Cantelli lemma. The lemma statement is as follows.

   Let $\{A_n, n \geq 1\}$ *be a sequence of events such that* $\sum_{n=1}^{\infty} P(A_n) < \infty$. *Then, almost surely, only finitely many of the* A_n*s will occur.*

   To prove this result, let us consider an arbitrary sequence of events $\{A_n, n \geq 1\}$, and define the nth "tail event"

   $$\Gamma_n = \bigcup_{i \geq n} A_i.$$

   (i) Give a verbal interpretation of the event Γ_n, in terms of the sequence of events $\{A_n, n \geq 1\}$.

   (ii) Consider next the event

   $$\bigcap_{n \geq 1} \Gamma_n = \bigcap_{n \geq 1} \bigcup_{i \geq n} A_i.$$

Explain why this event corresponds to the occurrence of infinitely many of the A_ns. *Hint:* Consider an $\omega \in \bigcap_{n\geq 1} \Gamma_n$, and argue why this ω is contained in infinitely many of the A_ns, and conversely.

(iii) Show that

$$P\left\{\bigcap_{n\geq 1}\bigcup_{i\geq n} A_i\right\} = \lim_{n\to\infty} P\left\{\bigcup_{i\geq n} A_i\right\}.$$

(iv) Suppose now that the sequence of events $\{A_n, n \geq 1\}$ satisfies $\sum_{n=1}^{\infty} P(A_n) < \infty$. Then, show that

$$\lim_{n\to\infty} P\left\{\bigcup_{i\geq n} A_i\right\} = 0.$$

Therefore, from parts (iv), (iii) and (ii), it follows that the event of infinitely many A_ns occurring has probability zero, implying the Borel-Cantelli lemma.

4. Exercise 5.6. (In this exercise, we find an explicit expression for $\{\omega : \lim_n Y_n(\omega) = 0\}$...)

   Note: This exercise shows the equivalence between the definition of almost sure convergence, and the equivalent condition that we stated as a theorem in class, but never proved. The exercise also shows that the subset of the sample space where convergence occurs is in fact a legitimate event, so that we can assign a probability measure to it.