EE6001: Stochastic Modeling and the theory of Queues

Tutorial 1

Due on 13/1/2011 by 4pm

1. Let A1 and A2 be arbitrary events and show that Pr{A1∪A2}+Pr{A1A2} = Pr{A1} + Pr{A2}. Explain which parts of the sample space are being double counted on both sides of this equation and which parts are being counted once.

2. This exercise derives the probability of an arbitrary (non-disjoint) union of events, derives the union bound, and derives some useful limit expressions.

   a) For 2 arbitrary events A_1 and A_2, show that$A_1 \cup A_2 = A_1 \cup (A_2 - A_1)$ where $A_2 - A_1 = A_2 {A_1}^c$. Show that A_1 and $A_2 - A_1$ are disjoint. Hint: This is what Venn diagrams were invented for.

   b) For an arbitrary sequence of events, {An; n ≥ 1}, let $B_1 = A_1$ and for each n ≥ 2 define $B_n = An - \bigcup_{m=1}^{n-1} A_m$. Show that $B_1, B_2, ...$, are disjoint events and show that for each n≥2, $\bigcup_{m=1}^{n} A_m = \bigcup_{m=1}^{n} B_m$. Hint: Use induction.

   c) Show that $Pr\{\bigcup_{n=1}^{\infty} A_n\} = Pr\{\bigcup_{n=1}^{\infty} B_n\} = \sum_{n=1}^{\infty} Pr\{B_n\}$

   Hint: Use the axioms of probability for the second equality.

   d) Show that for each n, $Pr\{B_n\} \leq Pr\{A_n\}$. Use this to show that $Pr\{\bigcup_{n=1}^{\infty} A_n\} \leq \sum_{n=1}^{\infty} Pr\{A_n\}$.

   e) Show that $Pr\{\bigcup_{n=1}^{\infty} A_n\} = lim_{m\to\infty} Pr\{\bigcup_{n=1}^{m} A_n\}$
   Hint: Combine parts c) and b). Note that this says that the probability of a limit of unions is equal to the limit of the probabilities. This might well appear to be obvious without a proof, but you will see situations later where similar appearing interchanges cannot be made.

   f) Show that $Pr\{\bigcap_{n=1}^{\infty} A_n\} = lim_{n\to\infty} Pr\{\bigcap_{i=1}^{n} A_i\}$
   Hint: Remember deMorgan's equalities.

3. This exercise shows that for all rv's X, $F_X(x)$ is continuous from the right.
   a) For any given rv X, any real number x, and each integer $n \geq 1$, let $A_n = \{\omega : X > x + 1/n\}$, and show that $A_1 \subseteq A_2 \subseteq .....$ Use this and the corollaries to the axiom of probability to show that
   $Pr\{\bigcup_{n\geq 1} A_n\} = lim_{n\to\infty} Pr\{A_n\}$.

   b) Show that $Pr\{\bigcup_{n\geq 1} A_n\} = Pr\{X > x\}$ and show that
   $Pr\{X > x\} = lim_{n\to\infty} Pr\{X > x + 1/n\}$.

   c) Show that for $\varepsilon > 0$, $lim_{\varepsilon\to 0} Pr\{X \leq x + \varepsilon\} = Pr\{X \leq x\}$.

   d) Define $\tilde{F}_X(x) = Pr\{X < x\}$. Show that $\tilde{F}_X(x)$ is continuous from the left. In other words, the continuity from the right for the distribution function arises from the almost arbitrary (but universal) choice in defining the distribution function as $Pr\{X \leq x\}$ rather than $Pr\{X < x\}$.

4. a) Can two disjoint events ever be independent?
   b) Can an event A be independent of itself?
   Justify your answer.

5. If you pick a real number from [0,1] according to a uniform distribution, what is the probability of the number being irrational? Is the event independent of itself?