Right Buffer Sizing Matters: Stability, Queuing Delay and Traffic Burstiness in Compound TCP

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Abstract—Motivated by recent concerns that queuing delays in the Internet are on the rise, we conduct a performance evaluation of Compound TCP in two topologies. The first topology consists of a single bottleneck, and the second consists of two distinct sets of flows, regulated by two edge routers, feeding into a core router. We consider long-lived flows, and a combination of long and short flows. For these models, we derive necessary and sufficient conditions for stability and prove that the dynamical system undergoes a Hopf bifurcation as the stability condition gets violated. Using a combination of analysis and packet-level simulations, we emphasise that larger buffers, in addition to increasing delay, are prone to inducing limit cycles. These limit cycles in turn induce synchronisation among TCP flows and tend to make the downstream traffic bursty.

I. INTRODUCTION

There is an increasing concern in the recent literature about the large and growing buffer sizes in the internet routers [4]. It has also been argued that queuing delays are on the rise [9] due to these large router buffers, which adversely affects the end-to-end performance. Motivated by these recent concerns, we conduct a performance evaluation of Compound TCP [14], with particular emphasis on the impact of buffer sizes on the overall performance.

End-to-end latency in the Internet is influenced by several factors, such as the size of router buffers, the flavour of TCP, and the choice of queue management scheme. For sizing router buffers, there are currently three regimes outlined in the literature [12]: a large, an intermediate, and a small buffer regime. In practice, today’s router buffers follow the traditional bandwidth-delay rule of thumb [15], which leads to larger buffers as the bandwidth increases. While numerous flavours of TCP have been proposed, Compound TCP [14] is the default protocol in the Windows operating system while Cubic TCP [6] is used in Linux. As far as queue management is concerned, sophisticated solutions, such as RED [2], have been proposed. However, a simple Drop-Tail policy which just drops incoming packets if the router buffer is full, is commonly used.

Our focus in this paper is on Compound TCP with Drop-Tail, in an intermediate and a small buffer regime. For completeness, we show that the models could also be used to analyse other flavours of TCP, such as Reno [10] and High-Speed TCP [3]. Currently, there are very few models for Compound which are amenable to an analytical performance evaluation. Recently, fluid models for a class of delay and loss-based protocols including Compound have been proposed [13], where the TCP window size is modelled using a non-linear, time-delayed dynamical system. Using such fluid models, we conduct a performance evaluation of Compound in two different topologies, under two buffer sizing regimes: an intermediate and a small buffer regime.

The first topology we consider is a single bottleneck link, which has either long-lived flows, or a combination of long-lived and short-lived flows. The second topology consists of two distinct sets of long-lived flows, regulated by two edge routers, each having a different round-trip time, and feeding into a core router. For both these topologies, we derive the necessary and sufficient conditions for local stability of the dynamical systems. For the single-link topology, we also derive the condition for non-oscillatory convergence. In each of these cases, a key insight obtained from our analysis is that smaller buffer thresholds are favourable for stability. Indeed, this insight holds even when short flows are present along with the long-lived flows. In addition, smaller buffers would also lead to reduced queuing delay, which is highly desirable.

Another important contribution of this paper lies in determining the behaviour of the system if it becomes unstable. In particular, we prove that the dynamical system undergoes a Hopf bifurcation (see [5]) under both topologies, when the stability condition is violated. The delay in the delay-differential equations describing the congestion avoidance phase of TCP plays an important role in maintaining the stability of the system. As the delay increases, the system undergoes a Hopf bifurcation when one pair of complex conjugate roots cross over the imaginary axis and lose stability. Since a Hopf bifurcation guarantees the emergence of limit cycles, our analysis, for both the models, predicts the emergence of limit cycles when the necessary and sufficient condition gets violated. These limit cycles induce synchronisation of the TCP windows, and cause periodic oscillations in the queue occupancy. In terms of performance, the emergence of limit cycles tends to make the downstream traffic bursty, and causes batch losses.

We confirm our analytical insights by conducting extensive packet-level simulations. Notably, the predictions of the fluid model agree remarkably well with the packet-level simulations even when with a moderate (about 60) number of long-lived flows in the system.

To summarise, our primary contribution is in highlighting the interplay between buffer thresholds and stability. In particular, larger queue thresholds would increase queuing delay, in addition to inducing limit cycles in the queue size dynamics. Such limit cycles can lead to a drop in link
utilisation, induce synchronisation among TCP flows, and make the downstream traffic bursty. Secondly, we find that even in the presence of short flows, limit cycles in the queue do not disappear. In other words, the ‘noise’ contributed by the short flows does not serve to desynchronise the limit cycles emerging due to the dynamics of the long flows.

The rest of this paper is organised as follows. In Section II, we briefly outline the congestion avoidance algorithms of Compound TCP. In Section III, we analyse a single bottleneck topology with either long-lived, or a combination of long and short flows. Our analytical results are corroborated by simulations, conducted at the packet-level, in the Network Simulator (NS2) [16]. In Section IV, we study the second topology using a combination of analysis and packet-level simulations. Finally, in Section V we summarise our contributions.

II. COMPOUND TCP

Compound TCP (C-TCP) [14] is a widely implemented Transmission Control Protocol (TCP) in the Windows operating system. Transport protocols like TCP Reno and High Speed TCP (HSTCP) use packet loss as the only indication of congestion. C-TCP is a synergy of both loss and delay-based feedback. The motivation behind incorporating both forms of feedback in C-TCP is to achieve high link utilisation and also to provide fairness to other competing TCP flows.

Compound TCP incorporates a scalable delay-based component into the congestion avoidance algorithm of TCP Reno. It controls its packet sending rate by maintaining two windows. The loss window \( cwnd \), behaves similarly as the loss window of TCP Reno whereas the delay window \( dwnd \), controls the delay-based component. In a time period of one round trip time, C-TCP updates its sending window \( w \) as follows:

\[
\begin{align*}
  w & = \min (cwnd + dwnd, awnd).
\end{align*}
\]

Here, \( awnd \) is the advertised window at the receiver side. In the congestion avoidance phase, the window size of Compound TCP evolves in the following manner.

\[
\begin{align*}
  w(t+1) = \begin{cases} 
    w(t) + \alpha w(t)^k & \text{if no loss} \\
    w(t)(1 - \beta) & \text{if loss}.
  \end{cases}
\end{align*}
\]

Here, \( \alpha, k \) are the increase parameters and \( \beta \) is the decrease parameter. The default values of these parameters are \( \alpha = 0.125 \), \( \beta = 0.5 \), and \( k = 0.75 \) [14].

III. SINGLE BOTTLENECK

This model consists of a single bottleneck link with many TCP flows feeding into a core router employing a Drop-Tail queue policy. The flows are subject to a common round trip time \( \tau \). Let the average window size of the flows be \( w(t) \). Then the average rate at which packets are sent is \( x(t) = w(t)/\tau \). Let the average congestion window increase by \( i(w(t)) \) for each received acknowledgement and decrease by \( d(w(t)) \) for each packet loss detected. The following nonlinear, time-delayed equation describes the behaviour of the average window size in the congestion avoidance phase [13]

\[
\begin{align*}
  \dot{w}(t) &= \frac{(i(w(t)) - d(w(t))p(t - \tau)) w(t - \tau)}{\tau}, \quad (1)
\end{align*}
\]

where \( p(t) \) denotes the loss probability experienced by packets sent at time \( t \). We analyse (1) in two scenarios: one, in which all the TCP flows are long-lived and another in which the traffic is a combination of both long and short flows. The loss probability \( p(t) \) depends on the average arrival rate, which in turn is related to the average window size. Then, (1) becomes

\[
\begin{align*}
  \dot{w}(t) &= \left( ii(w(t)) - d(w(t))p(w(t - \tau)) \right) \frac{w(t - \tau)}{\tau}. \quad (2)
\end{align*}
\]

The equilibrium of system (2) satisfies

\[
\begin{align*}
  i(w^*) = d(w^*)p(w^*). \quad (3)
\end{align*}
\]

Our focus will be on C-TCP, however other variants of TCP, like TCP Reno and High Speed TCP, can also be analysed via (2). The following are the functional forms of \( i(w(t)) \) and \( d(w(t)) \) for Compound, Reno and High Speed TCP.

- Compound TCP

\[
\begin{align*}
  i(w(t)) = \frac{\alpha w(t)^k}{w(t)} \quad \text{and} \quad d(w(t)) = \beta w(t). \quad (4)
\end{align*}
\]

- TCP Reno

\[
\begin{align*}
  i(w(t)) = \frac{1}{w(t)} \quad \text{and} \quad d(w(t)) = \frac{w(t)}{2}. \quad (5)
\end{align*}
\]

- High Speed TCP

\[
\begin{align*}
  i(w(t)) = \frac{f_1(w(t))}{w(t)} \quad \text{and} \quad d(w(t)) = f_2(w(t))w(t), \quad (6)
\end{align*}
\]

where \( f_1(\cdot) \) and \( f_2(\cdot) \) are continuous functions of the window size. We let \( u(t) = w(t) - w^* \), and linearise (2) about its non-trivial equilibrium point \( w^* \) to get

\[
\begin{align*}
  \dot{u}(t) &= -au(t) - bu(t - \tau), \quad (7)
\end{align*}
\]

where

\[
\begin{align*}
  a &= -\frac{w^*}{\tau}(i'(w^*) - d'(w^*)p(w^*)) \quad \text{and} \quad (8)
  b &= \frac{w^*}{\tau}p(w^*)d(w^*). \quad (9)
\end{align*}
\]

Looking for exponential solutions of (7), we get

\[
\begin{align*}
  \lambda + a + be^{-\lambda \tau} &= 0. \quad (10)
\end{align*}
\]
As stated in [11], if \( a \geq 0, b > 0, b > a \) and \( \tau > 0 \), a sufficient condition for stability of (7) is
\[
b\tau < \frac{\pi}{2},
\]
a necessary and sufficient condition for stability of (7) is
\[
\tau \sqrt{b^2 - a^2} < \cos^{-1}(-a/b),
\]
and the system undergoes the first Hopf bifurcation at
\[
\tau \sqrt{b^2 - a^2} = \cos^{-1}(-a/b).
\]

A. Local stability and Hopf bifurcation analysis

We now state conditions for local stability and Hopf bifurcation for Compound, Reno and High Speed TCP. To derive these conditions, we focus on long-lived flows. We consider the case where the core router has small buffer size and the system undergoes the first Hopf bifurcation at
\[
\tau \sqrt{b^2 - a^2} = \cos^{-1}(-a/b).
\]

Using condition (13), the associated Hopf condition can be stated. Clearly, the functional forms of \( f(w^*) \), \( d(w^*) \) and the model for the queue all greatly influence stability. Additionally, protocol parameters like \( \alpha \) and \( k \), and network parameters like queue thresholds or buffer sizes have to be chosen rather carefully if stability is to be ensured.

B. Non-oscillatory convergence with long-lived flows

Given these various conditions of stability, we now proceed to state a condition for non-oscillatory convergence of equation (7). For non-oscillatory convergence, we seek conditions on model parameters \( a, b \) and the delay \( \tau \) for which the characteristic equation (10) has negative real solution.

**Theorem 1:** The solution of the system shows non-oscillatory convergence if and only if the parameters \( a, b \) and \( \tau \) satisfy the condition \( \ln(b\tau) + a\tau + 1 < 0 \).

**Proof:** The boundary condition for the solution of (7) to be non-oscillatory is the point at which the curve \( f(\lambda) = \lambda + a + be^{-\lambda\tau} \) touches the real axis. If this point is \( \sigma \), then
\[
f(\sigma) = \sigma + a + be^{-\sigma \tau} = 0,
\]
\[
f'(\sigma) = 1 - b\tau e^{-\sigma \tau} = 0.
\]
From (21), we get
\[
be^{-\sigma \tau} = \frac{1}{\tau} \quad \text{and} \quad \sigma = \frac{\ln(b\tau)}{\tau}.
\]
Substituting values of \( \sigma \) and \( be^{-\sigma \tau} \) in (20) gives
\[
\ln(b\tau) + a\tau + 1 = 0.
\]
Then, a condition for non-oscillatory convergence of the equilibrium point is
\[
\ln(b\tau) + a\tau + 1 < 0.
\]

- **Compound TCP.** The condition for non-oscillatory convergence with Compound TCP is
\[
\frac{\alpha B}{\exp(\alpha (k - 2)(w^*)^{k-1} - 1)} < 1.
\]
We can immediately observe the relation between the protocol parameters \( \alpha \) and \( k \), and the network parameters like the buffer size \( B \) in ensuring that convergence of the equilibrium point is non-oscillatory.

- **TCP Reno.** The condition for TCP Reno is
\[
\frac{B}{w^* \exp(-\frac{2}{w^*} - 1)} < 1.
\]
This condition highlights the central role played by the parameter \( B \) to ensure non-oscillatory convergence.

- **High Speed TCP.** With High-Speed TCP flows, the corresponding condition is
\[
\frac{f_1(w^*) B}{w^* \exp\left( \frac{\mu (w^*)}{w^*} \left( \frac{w^* f_1'(w^*)}{f_1(w^*)} - 2 - \frac{\mu (w^*) f_1'(w^*) p(w^*)}{f_1(w^*)} \right) - 1 \right)} < 1.
\]
Long-lived flows

Fig. 2: Long-lived flows. 60 long-lived Compound TCP flows over a 2 Mbps link feeding into a single bottleneck queue with link capacity 100 Mbps.

Clearly functional forms of \( f_1(w^*) \) and \( f_2(w^*) \), and the choice of the parameter \( B \), all have a role to play. It is thus clear that both the design of transport protocols and queue management policies influence the dynamics of the system.

C. Long-lived and short-lived flows

We now deviate from the assumption that the system has only long-lived flows and consider the scenario where the traffic consists of both long and short flows. On a short time scale, short TCP connections may act as an uncontrolled and random background load on the network. Suppose the workload arriving at the bottleneck queue over a time period \( T \) is modelled as Gaussian with mean \( x^*T \) and variance \( x^*\sigma^2 T \) and the background load due to the short transfers over the time period \( T \) is also modelled as Gaussian with mean \( vT \) and variance \( v\sigma^2 T \). Then the loss probability at the bottleneck queue can be expressed as [8]

\[
p(w^*) = \exp \left( \frac{-2B (CT - w^* - v\tau)}{w^*\sigma^2_T + v\sigma^2_T} \right). \tag{28}
\]

Recall that (11) gives a sufficient condition for local stability and (13) gives the condition for which the system undergoes a Hopf-type bifurcation.

- **Compound TCP.** For Compound TCP, a sufficient condition for local stability of the system is

\[
2B\alpha (w^*)^k \frac{\nu\sigma^2_T + (C - v)\sigma^2_T}{(w^*\sigma^2_T + v\sigma^2_T)^2} < \frac{\pi}{2}. \tag{29}
\]

- **TCP Reno.** For TCP Reno, using the increase and decrease functions given by (5), we get a sufficient condition for stability as

\[
2B\alpha \frac{\nu\sigma^2_T + (C - v)\sigma^2_T}{(w^*\sigma^2_T + v\sigma^2_T)^2} < \frac{\pi}{2}. \tag{30}
\]

- **High Speed TCP.** With HSTCP flows, a sufficient condition for the local stability is

\[
2Bf_1(w^*) \frac{\nu\sigma^2_T + (C - v)\sigma^2_T}{(w^*\sigma^2_T + v\sigma^2_T)^2} < \frac{\pi}{2}. \tag{31}
\]

We are also in a position to state the Hopf bifurcation conditions, which are left out due to space constraints.

The conditions (29), (30) and (31) capture the relationships between the various protocol and network parameters. It is interesting to note that in general, larger the value of parameter \( B \), greater the possibility of driving the system to an unstable state. In Compound TCP, there appears to be an intrinsic trade off in the choice of the parameter \( \alpha \).
and the queue threshold parameter $B$. The presence of short-lived flows, which are modelled here as random uncontrolled traffic, does not change the requirement of choosing smaller values of $B$ to ensure stability.

### D. Simulations

We now conduct packet-level simulations, using NS2 [16], for the single bottleneck topology over an intermediate and a small buffer sizing regime. With small buffers, we employ 15 packets. With intermediate buffers, the buffer dimensioning rule is $B = C \cdot RTT/\sqrt{N}$, where $C$ is the bottleneck link capacity, $RTT$ is the average round trip time of TCP flows and $N$ is the number of long-lived flows in the system. Using this dimensioning rule, the bottleneck queue will have a buffer size of 270 packets. We consider two scenarios (i) only long-lived flows, and (ii) a combination of long and short flows. Qualitatively, the results seen in Fig. 3 depicts the simulation results where the system has a combination of long and short flows. The bottleneck link has a capacity of 100 Mbps. The packet size is fixed at 1500 bytes. In either scenario, there are 60 long-lived flows where each flow has an access link speed of 2 Mbps. The file size of each of the short flows is exponentially distributed with a mean file size of 1 Mb.

Fig. 2 depicts the simulations where the system only has long-lived flows. With small buffers, as expected, the queuing delay is negligible and the system is stable in the sense that there are no limit cycles in the queue size. With intermediate buffers, with smaller round trip times, the queues are full which yields full link utilisation but at the cost of extra latency. With larger delays, limit cycles will emerge in the queue size which also start to hurt link utilisation. Fig. 3 depicts the simulation results where the system has a combination of long and short flows. Qualitatively, the results are very similar to those shown in Fig. 2. This is expected as the models did indeed predict that despite the presence of short flows the system could readily lose stability if key system parameters were not properly dimensioned.

In the next section, we consider a multiple bottleneck topology where we will again consider the impact of long-lived, and a combination of long and short flows.

### IV. Multiple Bottlenecks

The model consists of two distinct sets of TCP flows having different round trip times $\tau_1$ and $\tau_2$ and regulated by two edge routers, as shown in Fig. 4. For this model, our focus will be on long-lived flows. The average window sizes of the two sets of flows are $w_1$ and $w_2$ respectively. The outgoing flows from both the edge routers feed into a common core router. The buffer sizes of the edge routers are $B_1$ and $B_2$ respectively, and buffer size of the core router is $B$. The link capacities of the edge routers are $C_1$ and $C_2$ respectively. We consider the case where both the edge routers and the core router have small buffer sizes and employ a Drop-Tail queuing policy. The link capacity of the core router is $C$. Suppose $p_1(t)$ and $p_2(t)$ are the packet loss probabilities at the two edge routers for the packets sent at time instant $t$, for the two distinct sets of flows respectively. The packet loss probability at the core router is denoted as $q(t, \tau_1, \tau_2)$. For a generalized TCP flavour, the non-linear, time-delayed, fluid model of the system is given by the following equations:

$$\dot{w}_j(t) = \frac{w_j(t - \tau_j)}{\tau_j} \left( i(w_j(t)) - d \left( ((w_j(t)) \left( p_j(t - \tau_j) + q(t, \tau_1, \tau_2) \right) \right) \right), j = 1, 2.$$

(32)

The loss probabilities at the three routers are approximated as

$$p_1(t) = \left( \frac{w_1(t)}{C_1 \tau_1} \right)^{B_1}, \quad p_2(t) = \left( \frac{w_2(t)}{C_2 \tau_2} \right)^{B_2}, \quad \text{and}$$

$$q(t, \tau_1, \tau_2) = \left( \frac{w_1(t - \tau_1)/\tau_1 + w_2(t - \tau_2)/\tau_2}{C} \right)^B.$$

The multiple bottleneck topology depicts a more realistic scenario. In this section, we prove that even in the multiple bottleneck topology, the system loses stability through a Hopf bifurcation [5] as system parameters vary. This loss of local stability leads to the emergence of limit cycles in the queue size of the core router.

#### A. Necessary and sufficient condition for stability

For system (32), we will perform a local stability analysis to derive a necessary and sufficient condition for stability. Suppose the equilibrium of the system is $(w_1^*, w_2^*)$. Let $u_1(t) = w_1(t) - w_1^*$ and $u_2(t) = w_2(t) - w_2^*$ be small perturbations about $w_1^*$ and $w_2^*$ respectively. Linearising system (32) about its equilibrium $(w_1^*, w_2^*)$, we get

$$\dot{u}_1(t) = -M_1 u_1(t) - N_1 u_1(t - \tau_1) - P_1 u_2(t - \tau_2),$$

$$\dot{u}_2(t) = -M_2 u_2(t) - N_2 u_2(t - \tau_2) - P_2 u_1(t - \tau_1),$$

(33)

where, for Compound TCP, the increase and decrease functions (4) yield the following coefficients

$$M_j = -\frac{A}{\tau_j} (k - 2) \left( \frac{w_j^*}{\tau_j} \right)^{k-1},$$

$$N_j = \frac{\beta B_j (w_j^*)^{B_j+1}}{(\tau_j)^{(B_j+1)}(C_j)^{B_j}} + \frac{\beta B (w_j^*)^2}{\tau_j^2 (C)^B} \left( \frac{w_1^*}{\tau_1} + \frac{w_2^*}{\tau_2} \right)^{B-1},$$

$$P_j = \frac{\beta B (w_j^*)^2}{\tau_1 \tau_2 (C)^B} \left( \frac{w_1^*}{\tau_1} + \frac{w_2^*}{\tau_2} \right)^{B-1}, \quad j = 1, 2.$$
At equilibrium, the following equations are satisfied
\[
\frac{\alpha}{\tau_j} (w_j^*)^{k-2} - \frac{\beta}{\tau_j} \left( \frac{w_j^*}{\tau_j C_j} \right)^B - \frac{\beta}{\tau_j C} \left( \frac{w_1^* + w_2^*}{\tau_1 + \tau_2} \right)^B = 0,
\]
\[j = 1, 2.\] For tractability, we assume that \( B_1 = B_2 = B, C_1 = C_2 = C, \tau_1 = \tau_2 = \tau. \) Then, \( w_1^* = w_2^* = w^* \) will be an equilibrium of the system, and satisfies the following equation:
\[
\alpha (w^*)^{k-2} = \beta (1 + 2B) \left( \frac{w^*}{\tau C} \right)^B.
\]

Let \( M = \frac{\beta B (w^*)^{B+1}}{1 + 2B}, \) then the coefficients \( M_1, M_2, N_1, N_2, P_1, P_2 \) reduce to
\[
M_1 = M_2 = -M \left( 1 + 2B \right) (k - 2) = a,
N_1 = N_2 = M (1 + 2B-1) = b,
P_1 = P_2 = M 2B-1 = c.
(34)

Note that \( a, b, c > 0. \) With these assumptions the linearised system (33) becomes
\[
\dot{u}_1(t) = -au_1(t) - bu_1(t - \tau) - cu_2(t - \tau),
\]
\[
\dot{u}_2(t) = -au_2(t) - bu_2(t - \tau) - cu_1(t - \tau).
(35)
\]

**Theorem 2:** The system (35) is stable if and only if the parameters \( a, b \) and \( c \) satisfy the condition \( \tau < \frac{1}{\omega_1} \cos^{-1} \left( \frac{-a}{b + c} \right) \)

with crossover frequency \( \omega_1 = \sqrt{(b + c)^2 - a^2}. \)

**Proof:** Looking for exponential solutions, we get the characteristic equation for the linearised system (35) as
\[
(\lambda + a + be^{-\lambda \tau})^2 - c^2 e^{-2\lambda \tau} = 0,
(36)
\]
which can be written as
\[
g_1(\lambda) g_2(\lambda) = 0,
\]
where,
\[
g_1(\lambda) = \lambda + a + (b + c) e^{-\lambda \tau}, \quad \text{and} \quad g_2(\lambda) = \lambda + a + (b - c) e^{-\lambda \tau}.
(37)
\]

For stability, all the roots of (36) should have negative real parts. Thus, the system becomes unstable if one pair of complex conjugate roots of either \( g_1(\lambda) \) or \( g_2(\lambda) \) or both crosses over the imaginary axis due to increase in the average delay \( \tau \), and hence have positive real parts. We find the points at which the roots of \( g_1(\lambda) \) and \( g_2(\lambda) \) cross over the imaginary axis. Substituting \( \lambda = j\omega_1 \) in \( g_1(\lambda) \) and separating real and imaginary parts we get
\[
(b + c) \sin \omega_1 \tau = \omega_1, \quad \text{and} \quad (b + c) \cos \omega_1 \tau = -a.
(38)
(39)
\]
Solving (38) and (39) for \( \omega_1 \) we get
\[
\omega_1 = \sqrt{(b + c)^2 - a^2},
\]
and under the condition \( b + c > a, \) a positive value of \( \omega_1^2 \) exists which implies that a value of \( \omega_1 \) exists at which the roots of the system having the characteristic equation \( g_1(\lambda) \) cross over and hence have positive real parts. Solving (38) and (39) for \( \tau, \) we get the critical value of delay at which the system transits from stability to instability as
\[
\tau_0 = \frac{1}{\omega_1} \cos^{-1} \left( \frac{-a}{b + c} \right).
(40)
\]
Similarly, substituting \( \lambda = j\omega_2 \) in \( g_2(\lambda) \) we get the crossover frequency as
\[
\omega_2 = \sqrt{(b - c)^2 - a^2}.
(41)
\]
We substitute the coefficients \( a, b \) and \( c \) in (41) to get
\[
\omega_2 = M \sqrt{\frac{1 - (1 + 2B)^2 (k - 2)^2}{B^2}}.
(42)
\]
It can be shown that, \( \omega_2 \) does not exist for any integer value of the buffer size \( B. \) Hence, the system having the characteristic equation \( g_2(\lambda) \) is stable for all values of the delay \( \tau. \) Hence, system (33) is asymptotically stable for all \( \tau < \tau_0, \) and unstable for \( \tau > \tau_0. \) Therefore, the necessary and sufficient condition for stability of (33) is
\[
\tau < \frac{1}{\omega_1} \cos^{-1} \left( \frac{-a}{b + c} \right).
(43)
\]
Substituting values of \( \omega_1, a, b \) and \( c \) in (43), we get the necessary and sufficient condition for local stability of (33) as
\[
\alpha (w^*)^{k-1} \sqrt{B^2 - (k - 2)^2} < \cos^{-1} \left( \frac{k - 2}{B} \right).
(44)
\]
This condition captures the relationship between the equilibrium window size, protocol parameters \( k, \) and \( \alpha, \) and buffer size \( B \) of the core router to ensure stability of the system. If we increase the buffer size of the core router, keeping other parameters fixed, the condition (44) would get violated.

**B. Hopf Condition**

We have seen that an increase in delay above the critical delay value prompts the system to transit from stability to instability. Varying the system parameters beyond the critical value can also drive the system to instability. Thus, instead of treating delay or any of the system parameters as the bifurcation parameter, we introduce an exogenous non-dimensional parameter \( \kappa \) which can act as the bifurcation parameter. If \( \kappa \) is varied keeping the values of the system parameters constant at their critical values, the system undergoes a Hopf bifurcation at \( \kappa = 1 \) and loses stability for further increase in \( \kappa. \) To show that the roots of the system (32) cross the imaginary axis with positive velocity as \( \kappa \) is varied and the system undergoes a Hopf Bifurcation, we proceed to verify the transversality condition of the Hopf spectrum [7].

Recall that \( u_1(t) = u_1(t) - w_1^* \) and \( u_2(t) = u_2(t) - w_2^*. \) The linearised system, with the non dimensional parameter
Looking for exponential solutions of (45) we get

$$\lambda = \frac{\kappa a e^{-\lambda \tau} + \kappa b e^{-\lambda \tau} - \kappa (b^2 - c^2) e^{-2\lambda \tau}}{\lambda + \kappa a + \kappa b e^{-\lambda \tau} - \kappa c e^{-\lambda \tau} - \kappa^2 (b^2 - c^2) e^{-2\lambda \tau}}. \tag{47}$$

Differentiating (46) with respect to $\kappa$, we get

$$\frac{d\lambda}{d\kappa} = -\kappa a^2 - \lambda a - \lambda b c e^{-\lambda \tau} - 2\kappa a b c e^{-\lambda \tau} - \kappa (b^2 - c^2) e^{-2\lambda \tau}. \tag{48}$$

From (46) we get,

$$e^{-\lambda \tau} = -\frac{\lambda + \kappa a}{\kappa (b + c)}. \tag{48}$$

$\kappa$ and the assumptions that $B_1 = B_2 = B, C_1 = C_2 = C, \tau_1 = \tau_2 = \tau$, now becomes

$$\dot{u}_1(t) = \kappa \left(-au_1(t) - bu_1(t - \tau) - cu_2(t - \tau)\right),$$

$$\dot{u}_2(t) = \kappa \left(-au_2(t) - bu_2(t - \tau) - cu_1(t - \tau)\right). \tag{45}$$

Looking for exponential solutions of (45) we get

$$(\lambda + \kappa a + \kappa (b + c) e^{-\lambda \tau}) (\lambda + \kappa a + \kappa (b - c) e^{-\lambda \tau}) = 0. \tag{46}$$

Fig. 5: Long-lived flows. Two sets of 60 long-lived Compound flows over a 2 Mbps link, regulated by two edge routers, feeding into a core router with link capacity 180 Mbps.

Fig. 6: Long-lived and short-lived flows. Two sets of 60 long-lived Compound flows over a 2 Mbps link, along with exponentially distributed short files, regulated by two edge routers and feeding into a core router with capacity 180 Mbps.
Next, Substituting (48) in (47), we get
\[
\frac{d\lambda}{d\kappa} = \frac{\lambda}{\kappa (1 + \lambda \tau + \kappa \tau_0)}.
\] (49)

At \(\tau = \tau_0, \kappa = \kappa_c\). Substituting \(\lambda = j\omega_1\) in (49) we get
\[
\text{Re} \left( \frac{d\lambda}{d\kappa} \right) = \frac{\omega_1^2 \tau_0}{\kappa_c \left( (1 + \kappa_c \alpha \tau_0)^2 + (\omega_1 \tau_0)^2 \right)} > 0.
\]

Thus the system undergoes a Hopf Bifurcation at \(\kappa = \kappa_c\). This implies that the system loses stability as the system parameters vary. This leads to the emergence of limit cycles. These limit cycles could in turn induce synchronisation among the Compound TCP flows which leads to periodic packet losses and makes the downstream traffic bursty.

C. Simulations

To validate our analytical insights, we simulate two scenarios in the multiple bottleneck topology: only long-lived flows, and a combination of long and short flows. The packet-level simulations are conducted in NS2 [16].

The system consists of two distinct sets of 60 Compound TCP flows, regulated by two edge routers, feeding into a common core router. The round trip time of one set of TCP flows is fixed at 10 ms, and that of the other set of flows is varied from 10 ms to 200 ms. The short-lived flows are exponentially distributed with a mean file size of 1Mb. Each edge router has a link capacity of 100 Mbps and the core router has a link capacity of 180 Mbps. In the small buffer regime, we fix the edge router buffers to 15 packets and the core router buffer size is varied from 15 packets to 100 packets. We recapitulate that in the intermediate buffer regime, the proposed buffer dimensioning rule is \(C : \frac{RTT}{\sqrt{N}}\). Using this buffer dimensioning rule, we fix the edge router buffers to be 270 packets and the core router buffer to be 360 packets.

Fig. 5 shows the simulations when there are only long-lived flows in the system. It is clear that larger queue threshold leads to non-linear oscillations, in the form of limit cycles, in the queue size. Such limit cycles can have a detrimental effect on link utilisation. Fig. 6 shows the simulations when there is a combination of long and short flows. Clearly, the time-delayed feedback effects of the long flows dominates the system stability and dynamics.

V. CONCLUDING REMARKS

In this paper, we conducted a performance evaluation of Compound TCP in two different topologies, under two buffer sizing regimes: an intermediate and a small buffer regime, with Drop-Tail queues. The key performance metrics considered were buffer thresholds and stability. For the traffic, we considered two cases: either just long-lived flows, or a combination of long and short flows.

Using a combination of analysis and packet-level simulations, we obtained some key insights. First, we emphasised the interplay between the buffer sizes and stability. In particular, we showed that larger queue policy thresholds may help link utilisation, but would increase queuing delay and are also prone to inducing limit cycles, via a Hopf bifurcation, in the queue size dynamics. However, such limit cycles can in turn lead to a drop in link utilisation, induce synchronisation among TCP flows, and make the downstream traffic bursty. Second, when a network has a combination of long and short flows, we highlight that despite the presence of short flows, such limit cycles continue to exist in the queue size. Some design considerations for protocol and network parameters, to ensure stability and low-latency queues, are also outlined.

There are still numerous questions that merit further investigation. For queue management, we focussed only on a Drop-Tail policy. It would be valuable to investigate the effect other queue policies, like RED [2], would have on queuing delay and stability, specially in multiple bottleneck topologies. We have repeatedly seen the existence of limit cycles in the queue size. We still need to address the asymptotic orbital stability of the bifurcating limit cycles in the multiple bottleneck topology which is the subject of a future publication.

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