1. (a) Prove that convergence in probability implies convergence in distribution, and give a counter-example to show that the converse need not hold.

(b) Show that convergence in distribution to a constant random variable implies convergence in probability to that constant.

2. Assume that a gambler’s winnings are determined by i.i.d fair coin tosses, as follows. At each stage $n$, his total wealth $Z_n$ is twice his previous wealth if the $n$th coin toss results in heads. The gambler is ruined (i.e., loses all his money) if the coin toss results in tails. What is his limiting expected wealth $\lim_n E[Z_n]$? What is the probability that the gambler is eventually ruined?

3. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables, such that $P\{X_n = 0\} = 1 - \frac{1}{n^2}$, and $P\{X_n = 1\} = \frac{1}{n^2}$.

   (i) Does $X_n$ converge to 0 in probability?
   (ii) Does $X_n$ converge to 0 almost surely?
   (iii) Does $X_n$ converge to 0 surely?
   (iv) Does $E[X_n]$ converge to 0?
   (v) Does $X_n$ converge to 0 in the mean-squared sense?
   (vi) Would any of your answers above change if the $X_n$s were dependent in some way, while maintaining the same marginals?

4. Let $\Omega = [0, 1]$ and let $P$ be the uniform probability measure on $\Omega$. Let $X_n$ be a sequence of random variables defined on this probability space such that

$$X_n(\omega) = \begin{cases} 1 & \text{if } \omega \in [0, 1/n], \\ 0 & \text{otherwise.} \end{cases}$$

   (i) Write down the PMF of $X_n$. Are the $X_n$s independent?
(ii) Does $X_n$ converge to 0 in probability?

(iii) Does $X_n$ converge to 0 almost surely? Surely?

5. Exercise 5.6, the one about mosquitoes.

6. Optional: Exercise 5.8 deals with the first Borel-Cantelli lemma, which would be useful to learn for those who have not seen it yet. See Grimmett and Stirzaker for the second lemma, which is a partial converse to the first. In particular, try reconciling your answers to Problem 3(ii), 4(iii), and the example dealt with in class, in light of the two lemmas.