30.1 Central Limit Theorem

In this section, we will state and prove the central limit theorem. Let \( \{X_i\} \) be a sequence of i.i.d. random variables having a finite variance. From law of large numbers we know that for large \( n \), the sum \( S_n \) is approximately as big as \( n\mathbb{E}[X] \), i.e.,

\[
\frac{S_n}{n} \xrightarrow{i.p.} \mathbb{E}[X],
\]

\[
\Rightarrow \frac{S_n - n\mathbb{E}[X]}{n} \xrightarrow{i.p.} 0.
\]

Thus whenever the variance of \( X_i \) is finite, the difference \( S_n - n\mathbb{E}[X] \) grows slower as compared to \( n \). The Central Limit Theorem (CLT) says that this difference scales as \( \sqrt{n} \), and that the distribution of \( \frac{S_n - n\mathbb{E}[X]}{\sqrt{n}} \) approaches a normal distribution as \( n \to \infty \) irrespective of the distribution of \( X_i \).

\[
\frac{S_n - n\mathbb{E}[X]}{\sqrt{n}} \sim N(0, \sigma_X^2).
\]

**Theorem 30.1 (Central Limit Theorem)** Let \( \{X_i\} \) be a sequence of i.i.d. random variables with mean \( \mathbb{E}[X] \) and a non-zero variance \( \sigma_X^2 < \infty \). Let \( Z_n = \frac{S_n - n\mathbb{E}[X]}{\sigma_X \sqrt{n}} \). Then, we have \( Z_n \overset{D}{\to} N(0,1) \), i.e.,

\[
\lim_{n \to \infty} F_{Z_n}(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx, \forall z \in \mathbb{R}.
\]

**Proof:** Let \( Y_n = \frac{X_n - \mathbb{E}[X]}{\sigma_X} \). Let \( Z_n = \frac{\sum Y_i}{\sqrt{n}} \). It is easy to see that \( Y_n \) has unit variance and zero mean, i.e., \( \mathbb{E}[Y_n] = 0 \) and \( \sigma_Y^2 = 1 \).

\[
C_{Y_n}(t) = 1 + it\mathbb{E}[Y_n] + \frac{t^2\mathbb{E}[Y_n^2]}{2} + O(t^2),
\]

\[
C_{Y_n}(t) = 1 + it(0) + \frac{t^2(1)^2}{2} + o(t^2),
\]

\[
= 1 - \frac{t^2}{2} + o(t^2),
\]

\[
C_{Z_n}(t) = \left[ C_{Y_n} \left( \frac{t}{\sqrt{n}} \right) \right]^n,
\]

\[
= \left[ 1 - \frac{t^2}{2n} + o \left( \frac{t^2}{n} \right) \right]^n \to e^{-\frac{t^2}{2}} \forall t.
\]
From the theorem on convergence of characteristic functions, $Z_n$ converges to a standard Gaussian in distribution.

For example, if $X_i$’s are discrete random variables, the CDFs will be step functions. As $n \to \infty$, these step functions will gradually converge to the error function (i.e. the steps will gradually decrease to form a continuous distribution as $n \to \infty$).

It is also important to understand what this theorem does not say. It is not saying that the probability density function converges to $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Convergence in density function requires more stringent conditions which are stated in the Local Central Limit Theorem.

**Theorem 30.2 (Local Central Limit Theorem)** Let $X_1, X_2, \ldots$ be i.i.d. random variables with zero mean and unit variance. Suppose further that their common characteristic function $\phi$ satisfies the following:

$$\int_{-\infty}^{\infty} |\phi(t)|^r dt < \infty,$$

for some integer $r \geq 1$. The density function $g_n$ of $U_n = \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}$ exists for $n \geq r$, and furthermore we have,

$$g_n(x) \to \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

as $n \to \infty$, uniformly in $x \in \mathbb{R}$.

**Proof:** For a proof, refer to Section 5.10 in [1].

Let $X_1, X_2, \ldots$ be i.i.d. random variables with zero mean and unit variance. From CLT, we know that $U_n = \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}}$ is distributed as a standard Gaussian. We now look at yet another interesting result which deals with the largest value taken by $U_m$, $m \geq n$, for a large $n$.

**Theorem 30.3 (The Law of the Iterated Logarithm)** Let $X_1, X_2, \ldots$ be i.i.d. random variables with zero mean and unit variance. Also, let $S_n = \sum_{i=1}^{n} X_i$ Then,

$$P \left( \limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1 \right) = 1.$$

Unlike the CLT which talks about distribution of $U_n$ for a large, fixed $n$, law of iterated logarithm talks about the largest fluctuation in $U_m$, for $m \geq n$. In particular, it bounds the largest value taken by $U_m$ beyond $n$. Formally, the subset of $\Omega$ for which this holds has a probability measure 1.

**References**