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# Lecture 22: Variance and Covariance

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In this lecture, we will introduce the notions of variance and covariance of random variables. Earlier, we learnt about the expected value of a random variable which gives an idea of the average value. The idea of variance is useful in describing the extent to which the random variable deviates about its mean on either side. The covariance is a property that characterizes the extent of dependence between two random variables.

## 22.1 Variance

As stated earlier, the variance quantifies the extent to which the random variable deviates about the mean. Mathematically, the variance is defined as follows :

**Definition 22.1** *Let $X$ be a random variable with $\mathbb{E}[X] < \infty$. The **variance** of $X$ is defined as*

$$Var(X) = \sigma_X^2 = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right].$$

*$\sigma_X$ is referred to as the **standard deviation** of the random variable $X$.*

### 22.1.1 Properties of Variance

We will now study a few properties of the variance of a random variable.

First and foremost, we can clearly see that for any real valued random variable $X$, $g(X) = (X - \mathbb{E}[X])^2 \geq 0$. Thus it is easy to see that $\sigma_X^2 \geq 0$ from property **PAI 2** from Lecture #18. In fact, we can make the following stronger statement regarding the variance of a random variable.

**Lemma 22.2** *Let $X$ be a real valued random variable. Then, $Var(X) = 0$ if and only if $X$ is a constant almost surely.*

**Proof:** We will first prove the sufficiency criterion in the above statement. That is, assume that $X$ is a constant valued random variable almost surely. Thus, it is evident that $X = \mathbb{E}[X]$ almost surely, consequently implying that $\sigma_X^2 = 0$.

To prove the necessity condition in the statement, assume that $X$ is a random variable with zero variance. Thus, we have the following :

$$\begin{aligned} \sigma_X^2 = \quad & \mathbb{E}[(X - \mathbb{E}[X])^2] &= 0. \\ \implies \quad & \int (X - \mathbb{E}[X])^2 d\mathbb{P}_X &= 0. \end{aligned} \tag{22.1}$$

Applying **PAI 7** from Lecture #18 to (22.1), we can conclude that $(X - \mathbb{E}[X])^2 = 0$ almost surely. Thus, we have $X = \mathbb{E}[X]$ almost surely.

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Now, using some simple algebra, we make a few useful observations.

$$\begin{aligned}
\sigma_X^2 &= \mathbb{E}\left[(X-\mathbb{E}[X])^2\right], \\
&= \mathbb{E}\left[(X^2)+(\mathbb{E}[X])^2-2X\mathbb{E}[X]\right], \\
&\overset{(a)}{=} \mathbb{E}\left[X^2\right]-2\mathbb{E}[X].\mathbb{E}[X]+(\mathbb{E}[X])^2, \\
&= \mathbb{E}\left[X^2\right]-(\mathbb{E}[X])^2, \qquad (22.2)
\end{aligned}$$

where $(a)$ follows from the linearity of expectation (**PAI 4** from Lecture #18). Now using the fact that $\sigma_X^2 \geq 0$ and (22.2), we can see that $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$. The term $\mathbb{E}\left[X^2\right]$ is referred to as the **second moment** of the random variable $X$.

An interesting digression:

**Theorem 22.3 (Jensen's Inequality)** *Let $X$ be any real valued random variable and let $h(\cdot)$ be a function of the random variable. Then,*

1. *If $h(\cdot)$ is convex, then $\mathbb{E}[h(X)] \geq h(\mathbb{E}[X])$.*
2. *If $h(\cdot)$ is concave, then $\mathbb{E}[h(X)] \leq h(\mathbb{E}[X])$.*
3. *If $h(\cdot)$ is linear, then $\mathbb{E}[h(X)] = h(\mathbb{E}[X])$.*

*A guided proof of Jensen's inequality will be encountered in your homework.*

Since $f(x) = x^2$ is a convex function, we can invoke Theorem 22.3 and observe that $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$.

Let us look at a few examples.

**Example 1:** Let $X$ be a Bernoulli random variable with parameter $p$ i.e.,

$$X = \begin{cases} 1 & \text{w.p. } p, \\ 0 & \text{w.p. } 1-p. \end{cases}$$

Find the variance of $X$.
**Solution:** We have

$$\begin{aligned}
\mathbb{E}[X] &= p \times 1 + (1-p) \times 0, \\
&= p.
\end{aligned}$$

Next,

$$\begin{aligned}
\mathbb{E}[X^2] &= p \times 1^2 + (1-p) \times 0^2, \\
&= p.
\end{aligned}$$

Finally,

$$\begin{aligned}
\sigma_X^2 &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2, \\
&= p - p^2, \\
&= p(1-p).
\end{aligned}$$

**Example 2:** Let $X$ be a discrete valued random variable with Poisson distribution of parameter $\lambda$. That is, $\mathbb{P}(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \forall k \in \mathbb{Z}^+ \cup \{0\}$. Find the variance of $X$.
**Solution:** We have

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=0}^{\infty} k \frac{e^{-\lambda}\lambda^k}{k!}, \\
&= \lambda \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^{(k-1)}}{(k-1)!}, \\
&= \lambda.
\end{aligned}$$

Next,

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{k=0}^{\infty} k^2 \frac{e^{-\lambda}\lambda^k}{k!}, \\
&= \sum_{k=1}^{\infty} \frac{\lambda(k-1+1)e^{-\lambda}\lambda^{(k-1)}}{(k-1)!}, \\
&= \sum_{k=2}^{\infty} \lambda^2 \frac{e^{-\lambda}\lambda^{(k-2)}}{(k-2)!} + \sum_{k=1}^{\infty} \lambda \frac{e^{-\lambda}\lambda^{(k-1)}}{(k-1)!}, \\
&= \lambda^2 + \lambda.
\end{aligned}$$

Finally,

$$\begin{aligned}
\sigma_X^2 &= \mathbb{E}[X^2] - \left(\mathbb{E}[X]\right)^2, \\
&= \lambda^2 + \lambda - (\lambda)^2, \\
&= \lambda.
\end{aligned}$$

**Example 3:** Let $X$ be a discrete random variable with $\mathbb{P}(X = k) = \frac{1}{\zeta(3)}\frac{1}{k^3}$ for $k \in \mathbb{N}$, where $\zeta(\cdot)$ is the Riemann zeta function. Find $\sigma_X^2$.
**Solution:** We have,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=1}^{\infty} k\mathbb{P}(X = k), \\
&= \sum_{k=1}^{\infty} k \frac{1}{\zeta(3)} \frac{1}{k^3}, \\
&= \frac{1}{\zeta(3)} \sum_{k=1}^{\infty} \frac{1}{k^2}, \\
&= \frac{1}{\zeta(3)} \frac{\pi^2}{6}.
\end{aligned}$$

Next, we have

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{k=1}^{\infty} k^2 \mathbb{P}(X = k), \\
&= \sum_{k=1}^{\infty} k^2 \frac{1}{\zeta(3)} \frac{1}{k^3}, \\
&= \frac{1}{\zeta(3)} \sum_{k=1}^{\infty} \frac{1}{k}, \\
&= \infty.
\end{aligned}$$

Finally,

$$\begin{aligned}
\sigma_X^2 &= \mathbb{E}\left[X^2\right] - \left(\mathbb{E}[X]\right)^2, \\
&= \infty.
\end{aligned}$$

The above example is a case of a random variable with finite expected value but infinite variance!

**Example 4:** Let $X$ be a uniform random variable in the interval $[a, b]$. Find the variance of $X$.
**Solution:** Recall that the density of $X$ is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b, \\ 0 & \text{otherwise .} \end{cases}$$

Now, we have

$$\begin{aligned}
\mathbb{E}[X] &= \int x f_X(x) dx, \\
&= \int_a^b x \, \frac{1}{b-a} dx, \\
&= \frac{a+b}{2}.
\end{aligned}$$

Next, we have

$$\begin{aligned}
\mathbb{E}\left[X^2\right] &= \int x^2 f_X(x) dx, \\
&= \int_a^b x^2 \, \frac{1}{b-a} dx, \\
&= \frac{(b^3 - a^3)}{3(b-a)}, \\
&= \frac{a^2 + ab + b^2}{3}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\sigma_X^2 &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2, \\
&= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}, \\
&= \frac{b^2-2ab+a^2}{12}, \\
&= \frac{(b-a)^2}{12}.
\end{aligned}$$

**Example 5:** Let $X$ be an exponentially distributed random variable with parameter $\mu$. Find $\sigma_X^2$.
**Solution:** Recall that for an exponential random parameter $f_X(x) = \mu e^{-\mu x}$ for $x \geq 0$.

$$\begin{aligned}
\mathbb{E}[X] &= \int x f_X(x) dx, \\
&= \int_0^\infty x \ \mu e^{-\mu x} dx, \\
&= \frac{1}{\mu}.
\end{aligned}$$

Next, we have

$$\begin{aligned}
\mathbb{E}\left[X^2\right] &= \int x^2 f_X(x) dx, \\
&= \int_0^\infty x^2 \ \mu e^{-\mu x} dx, \\
&= \frac{2}{\mu^2}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\sigma_X^2 &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2, \\
&= \frac{2}{\mu^2} - \left(\frac{1}{\mu}\right)^2, \\
&= \frac{1}{\mu^2}.
\end{aligned}$$

**Example 6:** Let $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$. Find $\sigma_X^2$.
**Solution:** From Example 2 in Lecture #21, we know that $\mathbb{E}[X] = \mu$ and $\mathbb{E}\left[X^2\right] = \mu^2 + \sigma^2$. Thus, we have

$$\begin{aligned}
\sigma_X^2 &= \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2, \\
&= \mu^2 + \sigma^2 - (\mu)^2, \\
&= \sigma^2.
\end{aligned}$$

Note that the normal distribution is parametrized by the expected value $\mu$ and the variance $\sigma^2$.

## 22.2 Covariance

Having looked at variance, a term that characterizes the extent of deviation of a single random variable around its expected value, we now define and study the covariance of two random variables $X$ and $Y$, a term that quantifies the extent of dependence between the two random variables.

**Definition 22.4** *Let $X$ and $Y$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Further, let $\mathbb{E}[X] < \infty$ and $\mathbb{E}[Y] < \infty$. The **covariance** of $X$ and $Y$ is given by*

$$cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

**Definition 22.5** *Let $X$ and $Y$ be random variables. $X$ and $Y$ are said to be **uncorrelated** if $cov(X, Y) = 0$, i.e, if $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.*

Thus, two random variables are uncorrelated if the expectation of their product is the product of their expectations. The following theorem asserts that independent random variables are uncorrelated.

**Theorem 22.6** *If $X$ and $Y$ are independent random variables with $\mathbb{E}[|X|] < \infty$, $\mathbb{E}[|Y|] < \infty$. Then $\mathbb{E}[XY]$ exists, and $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ i.e., $cov(X, Y) = 0$.*

**Proof:** We will prove this theorem in three steps. We will first assume that the random variables $X$ and $Y$ are simple and thus can be represented as follows :

$$X = \sum_{i=1}^{n} x_i \mathbb{I}_{A_i} \quad \text{and} \quad Y = \sum_{i=1}^{m} y_i \mathbb{I}_{B_i}.$$

Assuming canonical representation of the random variables $X$ and $Y$, we have

$$XY = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i y_j) \mathbb{I}_{(A_i \cap B_j)}.$$

Thus, we have,

$$\begin{aligned} \mathbb{E}[XY] &= \int XY d\mathbb{P}, \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i y_j) \mathbb{P}(A_i \cap B_j). \end{aligned} \tag{22.3}$$

Next, as $X$ and $Y$ are independent random variables, $\sigma(X)$ and $\sigma(Y)$ are independent $\sigma-$algebras. Also, $A_i = \{\omega \in \Omega | X(\omega) = a_i\} \in \sigma(X)$ and $B_j = \{\omega \in \Omega | X(\omega) = b_j\} \in \sigma(Y)$. By definition of independent $\sigma-$algebras,

$$\mathbb{P}(A_i \cap B_j) = \mathbb{P}(A_i)\mathbb{P}(B_j), \ \forall i, j. \tag{22.4}$$

Using (22.4) in (22.3), we get

$$\begin{aligned} \mathbb{E}[XY] &= \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j \mathbb{P}(A_i)\mathbb{P}(B_j), \\ &= \left( \sum_{i=1}^{n} x_i \mathbb{P}(A_i) \right) \left( \sum_{j=1}^{m} y_j \mathbb{P}(B_j) \right), \\ &= \mathbb{E}[X]\mathbb{E}[Y]. \end{aligned}$$

We will now extend the proof to non-negative random variables. Let $X$ and $Y$ be non-negative random variables. Let the sequences of simple random variables, $X_n$ and $Y_n$, be such that $X_n \uparrow X$ and $Y_n \uparrow Y$. We know that such a sequence exists from section 3 in Lecture #19. Also, by construction, it is easy to see that $X_n$ and $Y_n$ are independent. Consequently, we have $X_nY_n \uparrow XY$. Thus,

$$\mathbb{E}[XY] \overset{MCT}{=} \lim_{n\to\infty} \mathbb{E}[X_nY_n] \overset{(a)}{=} \left(\lim_{n\to\infty} \mathbb{E}[X_n]\right)\left(\lim_{n\to\infty} \mathbb{E}[Y_n]\right) \overset{MCT}{=} \mathbb{E}[X]\mathbb{E}[Y], \tag{22.5}$$

where $(a)$ follows from the independence of $X_n$ and $Y_n$ and since both the limits exist.

Finally, for the case of $X$ and $Y$ possibly being negative, let $X = X_+ - X_-$, and let $Y = Y_+ - Y_-$ where $X_+, X_-, Y_+$ and $Y_-$ are as defined in Lecture #17. Then

$$\begin{aligned}
\mathbb{E}[XY] &= \mathbb{E}[X_+Y_+] + \mathbb{E}[X_-Y_-] - \mathbb{E}[X_+Y_-] - \mathbb{E}[X_-Y_+], && (22.6)\\
&= \mathbb{E}[X_+]\mathbb{E}[Y_+] + \mathbb{E}[X_-]\mathbb{E}[Y_-] - \mathbb{E}[X_+]\mathbb{E}[Y_-] - \mathbb{E}[X_-]\mathbb{E}[Y_+], && (22.7)\\
&= \mathbb{E}[X]\mathbb{E}[Y]. && (22.8)
\end{aligned}$$

where (22.6) and (22.8) follow from the linearity of expectations (**PAI 4** from Lecture #18) and (22.7) follows from (22.5). Note that $X_+$ and $X_-$ are functions of $X$, and $Y_+$ and $Y_-$ are functions of $Y$. Since $X$ and $Y$ are independent, all the pairs of random variables inside expectation in RHS of (22.6) are independent.[1] Thus, we have proved that independent random variables are uncorrelated.

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**Caution:** While independence guarantees that two random variables are uncorrelated, the converse is not necessarily true i.e., two uncorrelated random variables may or may not be independent. We show this by a counter example.

Let $X \sim unif[-1, 1]$ and $Y = X^2$ be two random variables. It can be shown that $X$ and $Y$ are not independent. However,

$$\begin{aligned}
cov(X, Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y],\\
&= \mathbb{E}\left[X^3\right] - \mathbb{E}[X]\mathbb{E}\left[X^2\right],\\
&\overset{(a)}{=} 0 - 0,\\
&= 0,
\end{aligned}$$

where $(a)$ follows since $X$ is symmetric around 0.
Thus, we have an example where two random variables $X$ and $Y$ are uncorrelated but not independent.

**Proposition 22.7** *Consider two random variables $X$ and $Y$. Then, we have*

$$Var(X + Y) = Var(X) + Var(Y) + 2cov(X, Y).$$

**Proof:**

$$\begin{aligned}
Var(X+Y) &= \mathbb{E}\left[(X+Y)^2\right] - (\mathbb{E}[X] + \mathbb{E}[Y])^2\\
&= \mathbb{E}\left[X^2 + Y^2 + 2XY\right] - \left(\mathbb{E}[X]^2 + \mathbb{E}[Y]^2 + 2\mathbb{E}[X]\mathbb{E}[Y]\right),\\
&= \left(\mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2\right) + \left(\mathbb{E}\left[Y^2\right] - \mathbb{E}[Y]^2\right) + 2\left(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\right),\\
&= Var(X) + Var(Y) + 2cov(X, Y).
\end{aligned}$$

[1] Let $X$ and $Y$ be independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Also, let $f(\cdot)$ and $g(\cdot)$ be measurable functions from $\mathbb{R}$ to $\mathbb{R}$. Then, $f(X)$ and $g(Y)$ are independent random variables.

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It is easy to see that if $X$ and $Y$ are uncorrelated, then $Var(X+Y) = Var(X) + Var(Y)$. This can of course be extended to the sum of any finite number of random variables.

**Definition 22.8** *Let $X$ and $Y$ be random variables. Then, the* ***correlation coefficient*** *for the two random variables is defined as :*

$$\rho_{X,Y} \triangleq \frac{cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

**Theorem 22.9** ***Cauchy-Schwartz Inequality*** *For any two random variables $X$ and $Y$, $-1 \leq \rho_{X,Y} \leq 1$. Further, if $\rho_{X,Y} = 1$, then there exists $a > 0$ such that $Y - \mathbb{E}[Y] = a\,(X - \mathbb{E}[X])$ and if $\rho_{X,Y} = -1$, then there exists $a < 0$ such that $Y - \mathbb{E}[Y] = a\,(X - \mathbb{E}[X])$.*

**Proof:** Let $\widetilde{X} = X - \mathbb{E}[X]$ and $\widetilde{Y} = Y - \mathbb{E}[Y]$. Now we know that,

$$\mathbb{E}\left[\left(\widetilde{X} - \frac{\mathbb{E}[\widetilde{X}\widetilde{Y}]}{\mathbb{E}[\widetilde{Y}^2]}\widetilde{Y}\right)^2\right] \overset{(a)}{\geq} 0, \tag{22.9}$$

$$\mathbb{E}\left[\widetilde{X}^2 - 2\widetilde{X}\frac{\mathbb{E}[\widetilde{X}\widetilde{Y}]}{\mathbb{E}[\widetilde{Y}^2]}\widetilde{Y} + \frac{\left(\mathbb{E}[\widetilde{X}\widetilde{Y}]\right)^2}{\left(\mathbb{E}[\widetilde{Y}^2]\right)^2}\widetilde{Y}^2\right] \geq 0,$$

$$\mathbb{E}\left[\widetilde{X}^2\right] - \frac{(\mathbb{E}[\widetilde{X}\widetilde{Y}])^2}{\mathbb{E}[\widetilde{Y}^2]} \overset{(b)}{\geq} 0,$$

$$\mathbb{E}\left[\widetilde{X}^2\right] \geq \frac{(\mathbb{E}[\widetilde{X}\widetilde{Y}])^2}{\mathbb{E}[\widetilde{Y}^2]},$$

$$-1 \leq \frac{\mathbb{E}\left[\widetilde{X}\widetilde{Y}\right]}{\sqrt{\mathbb{E}\left[\widetilde{X}^2\right]}\sqrt{\mathbb{E}\left[\widetilde{Y}^2\right]}} \leq 1, \tag{22.10}$$

where $(a)$ follows from **PAI 2** of Lecture #18 and $(b)$ follows from linearity and scaling property of expectation (**PAI 4** and **PAI 8** of Lecture #18). From definition, $\mathbb{E}[\widetilde{X}^2] = Var(X)$ and $\mathbb{E}[\widetilde{Y}^2] = Var(Y)$. Further, we can observe that $\mathbb{E}[\widetilde{X}\widetilde{Y}] = cov(X,Y)$.Thus, it is easy to see that

$$\rho_{X,Y} = \frac{cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{\mathbb{E}\left[\widetilde{X}\widetilde{Y}\right]}{\sqrt{\mathbb{E}\left[\widetilde{X}^2\right]}\sqrt{\mathbb{E}\left[\widetilde{Y}^2\right]}}\ . \tag{22.11}$$

Combining (22.10) and (22.11), we get

$$-1 \leq \rho_{X,Y} \leq 1.$$

Note that $\rho_{X,Y} = 1$ or $\rho_{X,Y} = -1$ when the (22.9) is met with equality. This happens when $\widetilde{X} = \frac{\mathbb{E}[\widetilde{X}\widetilde{Y}]}{\mathbb{E}[\widetilde{Y}^2]}\widetilde{Y}$ almost surely which proves the second part of the theorem.

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The discussion regarding Cauchy-Schwartz inequality above has a close connection with Hilbert Spaces. As one may recall from a course in Linear Algebra, a Hilbert Space is a complete vector space endowed with an inner product.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{L}_2$ be a collection of all zero-mean, real-valued random variables defined over this space with finite second moment. It can be shown that $\mathcal{L}_2$ with addition of functions and scalar multiplication (obeyed except perhaps on a set of measure zero) is a Hilbert Space. The associated inner product is the covariance function. We say that two random variables from $\mathcal{L}_2$ are *equivalent* if they agree, except perhaps on a set of measure zero. That is, $X \sim Y$ (read as $X$ is equivalent to $Y$) if $\mathbb{P}(X = Y) = 1$, for any $X, Y \in \mathcal{L}_2$. Thus, $\mathcal{L}_2$ is partitioned into several such equivalence classes by the aforementioned equivalence relation.

In light of this discussion, the covariance function can be interpreted as the dot product of the Hilbert space, and the correlation coefficient is interpreted as the cosine of the angle between two random variables in this Hilbert space. In particular uncorrelated random variables are orthogonal! The interested reader is referred to sections 7 through 11 of chapter 6 in [1] for a more detailed treatment of this topic; this viewpoint is especially useful in estimation theory.

## 22.3 Exercise

1. *[Papoulis]* Let $a$ and $b$ be positive integers with $a \leq b$, and let $X$ be a random variable that takes as values, with equal probability, the powers of 2 in the interval $[2^a, 2^b]$. Find the expected value and variance of $X$.

2. *[Papoulis]* Suppose that $X$ and $Y$ are random variables with the same variance. Show that $X - Y$ and $X + Y$ are uncorrelated.

3. *[Papoulis]* Suppose that a random variable $X$ satisfies $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] = 1$, $\mathbb{E}[X^3] = 0$ and $\mathbb{E}[X^4] = 3$ and let $Y = a + bX + cX^2$. Find the correlation co-efficient $\rho_{X,Y}$.

4. *[Assignment problem, University of Cambridge]* Take $0 \leq r \leq 1$. Let $X$ and $Y$ be independent random variables taking values $\pm 1$ with probabilities $\frac{1}{2}$. Set $Z = X$, with probability $r$ and $Z = Y$, with probability $1 - r$. Find $\rho_{X,Z}$.

5. *[Papoulis]* Let $X_1, X_2, ..., X_n$ be independent random variables with non-zero finite expectations. Show that
$$\frac{\operatorname{var}(\prod_{i=1}^{n} X_i)}{\prod_{i=1}^{n} \mathbb{E}[X_i]^2} = \prod_{i=1}^{n} \Big(\frac{\operatorname{var}(X_i)}{\mathbb{E}[X_i]^2} + 1\Big) - 1$$
.

## References

[1] David Williams, "Probability with Martingales", *Cambridge University Press*, Fourteenth Printing, 2011.