EE5110: Probability Foundations for Electrical Engineers **July-November 2015**

Lecture 18: Properties of Abstract Integrals

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In this lecture, we discuss some basic properties of abstract integrals.

18.1 Properties of Abstract Integrals

We will state the properties for a generic abstract integral, and also particularize for the special case of the expectation of a random variable.

Let $(\Omega, \mathcal{F})$ be a measurable space and f, g, h be measurable functions from Ω to $\mathbb{R}$. Let μ be a generic measure and $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$. Let X, Y be random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. The first part of each property corresponds to the generic measure (μ), while the second part particularizes to the probability measure $(\mathbb{P})$.

In order to prove the properties, we follow a procedure: we begin by proving them for simple functions, and then extend it for the case of non-negative functions. Finally using these, we prove the properties for general measurable functions.

[PAI 1] $\int \mathbb{I}_A \, d\mu = \mu(A)$, for any $A \in \mathcal{F}$. In particular, for any $A \in \mathcal{F}$, we have $\mathbb{E}[\mathbb{I}_A] = \mathbb{P}(A)$.

Proof: $f(\omega) = I_A(\omega)$ is a simple function and is in canonical form. So, proof directly follows from the definition of integral of simple functions (Definition 17.4 from Lecture #17).

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[PAI 2] If $g \geq 0$, then $\int g \, d\mu \geq 0$. If $X \geq 0$, then $\mathbb{E}(X) \geq 0$.

Proof: Let g be a simple function and $g \geq 0$. A simple function is of the form equation (17.1) from Lecture #17 with all a_i's non-negative. So, $\int g \, d\mu \geq 0$.

Now, we prove the property for non-negative functions. Let g be a non-negative function. Let $\mathcal{S}(g)$ contains all simple functions, $q(\omega)$ such that $q(\omega) \leq g(\omega), \ \forall \omega \in \Omega$.
So, $\int q \, d\mu \geq 0, \ \forall q \in \mathcal{S}(g)$, since q is a simple function.
Hence, $\int g \, d\mu = \sup\limits_{q \in \mathcal{S}(g)} \int q d\mu \geq 0$, since supremum of a set of non-negative numbers is non-negative.

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[PAI 3] If $g = 0$, μ.a.e., then $\int g \, d\mu = 0$. If $X = 0$ a.s., then $\mathbb{E}(X) = 0$.

Proof: Let $g = 0$ μ.a.e. be a simple function. Then, g has a canonical representation of the form $g = \sum_{i=1}^{k} a_i \mathbb{I}_{A_i}$, where $\mu(A_i) = 0$, for each i. Hence, $\int g \, d\mu = 0$.

Let $g = 0$ μ.a.e. be a non-negative function. Let $q \in \mathcal{S}(g) \Rightarrow q(\omega) \leq g(\omega), q(\omega) \geq 0, \ \forall q \in \mathcal{S}(g)$ and $\forall \omega$.
Since, $g = 0$ we have $q(\omega) = 0, \ \forall \omega$ i.e., $q(\omega) = 0$, μ.a.e..
Hence, $\int q \, d\mu = 0, \ \forall q \in \mathcal{S}(g) \Rightarrow \int g \, d\mu = 0$.

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[PAI 4] **[Linearity]** $\int (g + h) \, d\mu = \int g \, d\mu + \int h \, d\mu$. And, $\mathbb{E}(X + Y) = E(X) + E(Y)$.

Proof: Let g and h be simple functions. We can write g and h in canonical representation form as:

$$g = \sum_{i=1}^{k} a_i \mathbb{I}_{A_i}, \qquad h = \sum_{j=1}^{m} b_j \mathbb{I}_{B_j},$$

where the sets A_i are disjoint, and the sets B_j are also disjoint. So, the sets $A_i \cap B_j$ are disjoint. Then, $g + h$ can be written as:

$$g + h = \sum_{i=1}^{k} \sum_{j=1}^{m} (a_i + b_j) \mathbb{I}_{A_i \cap B_j}.$$

So,

$$\begin{aligned}
\int (g+h)\, d\mu &\overset{(a)}{=} \sum_{i=1}^{k} \sum_{j=1}^{m} (a_i + b_j)\, \mu(A_i \cap B_j), \\
&= \sum_{i=1}^{k} a_i \sum_{j=1}^{m} \mu(A_i \cap B_j) + \sum_{j=1}^{m} b_j \sum_{i=1}^{k} \mu(A_i \cap B_j), \\
&\overset{(b)}{=} \sum_{i=1}^{k} a_i \mu(A_i) + \sum_{j=1}^{m} b_j \mu(B_j), \\
&\overset{(c)}{=} \int g\, d\mu + \int h\, d\mu.
\end{aligned}$$

Where (a) and (c) are due to definition of integral of simple functions, (b) is due to finite additivity of μ.

Proving linearity for general non-negative functions is not easy at this point. We will return to finish this proof after equipping ourselves with the Monotone Convergence Theorem.

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[PAI 5] If $0 \le g \le h$ μ.a.e., then $\int g\, d\mu \le \int h\, d\mu$. In particular, if $0 \le X \le Y$ a.s., then $\mathbb{E}(X) \le \mathbb{E}(Y)$.

Proof: Let g and h be simple functions and $0 \le g \le h$ μ.a.e.. Then, we have $h = g + q$, for some simple function $q \ge 0$ μ.a.e.. But, we can write $q = q_+ - q_-$, where $q_+ \ge 0$ and $q_- \ge 0$, and $q_- = 0$ μ.a.e.. Here, q, q_+, q_- are all simple functions. Using the linearity property [PAI 4] and then properties [PAI 3], [PAI 2], we write

$$\int h\, d\mu = \int g\, d\mu + \int q_+\, d\mu - \int q_-\, d\mu = \int g\, d\mu + \int q_+\, d\mu \ge \int g\, d\mu.$$

Let g and h be non-negative functions and $0 \le g \le h$ μ.a.e.. Let $q \in \mathcal{S}(g) \Rightarrow q(\omega) \le g(w) \le h(\omega)\, \forall\, \omega \in \Omega \Rightarrow q \in \mathcal{S}(h)$. So, $\mathcal{S}(g) \subseteq \mathcal{S}(h)$. Hence, $\sup\limits_{q \in \mathcal{S}(g)} \int q\, d\mu \le \sup\limits_{q \in \mathcal{S}(h)} \int q d\mu \Rightarrow \int g\, d\mu \le \int h\, d\mu$.

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[PAI 6] If $g = h$ μ.a.e., then $\int g\, d\mu = \int h\, d\mu$. If $X = Y$ a.s., then $\mathbb{E}(X) = \mathbb{E}(Y)$.

Proof: The proof follows from the above property since $g = h$ μ.a.e. $\Leftrightarrow g \le h$ μ.a.e. and $h \le g$ μ.a.e.

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Example: The Dirichlet function and the zero function are equal μ.a.e.. So, they have the same integral equal to zero under Lebesgue measure.
Note that the measure with respect to which the integration is performed on both sides must be the same for the equality to hold.

[PAI 7] If $g \geq 0$ μ.a.e., and $\int g\, d\mu = 0$, then $g = 0$ μ.a.e.. If $X \geq 0$ a.s., and $\mathbb{E}(X) = 0$, then $X = 0$ a.s..

Proof: Let g be a simple function. Let $g \geq 0$ μ.a.e., and $\int g\, d\mu = 0$. We can write $g = g_+ - g_-$, where $g_+ \geq 0$ and $g_- \geq 0$. Then, $g_- = 0$ μ.a.e.. Using [PAI 3], we get $\int g_-\, d\mu = 0$.
Due to Linearity property[PAI 4], we can write $\int g_+\, d\mu = \int g\, d\mu + \int g_-\, d\mu = 0$.

Observe that g_+ is a simple function. So, g_+ has a canonical representation of the form: $g_+ = \sum\limits_{i=1}^{k} a_i \mathbb{I}_{A_i}$, with $a_i > 0$ for each i. It follows that $\mu(A_i) = 0\, \forall\, i$, since $\sum\limits_{i=1}^{k} a_i \mu(A_i) = 0$. Due to finite additivity, we conclude that $\mu\left(\bigcup\limits_{i=1}^{k} A_i\right) = 0$. Therefore, $g_+ = 0$ μ.a.e., and $g = 0$ μ.a.e..

Let g be a non-negative function. We use proof by contradiction method to prove this property.
Suppose the contrary, i.e., $B = \{\omega | g(\omega) > 0\}$, where $\mu(B) > 0$.
Let $B_n = \{\omega | g(\omega) > \frac{1}{n}\}$. Clearly $B_n \subseteq B_{n+1}\, \forall n \in \mathbb{N}$ and $\bigcup\limits_{i=1}^{\infty} B_n = B$. So, $\mu(B) = \mu\left(\bigcup\limits_{i=1}^{\infty} B_n\right) = \lim\limits_{n\to\infty} \mu\left(B_n\right) > 0$ which implies that $\exists\, k \in \mathbb{N}$ such that $\mu\left(B_k\right) > 0$ (from properties of limits of sequences).
$\int g\, d\mu \geq \int\limits_B g\, d\mu = \int g \mathbb{I}_B\, d\mu \geq \int g \mathbb{I}_{B_k}\, d\mu > \int \frac{1}{k} \mathbb{I}_{B_k}\, d\mu = \frac{1}{k}\mu\left(B_k\right) > 0$, which is a contradiction!

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[PAI 8] **[Scaling]** Let $a \geq 0$. Then $\int (af)\, d\mu = a \int f d\mu$. If $a \geq 0$, then $\mathbb{E}(aX) = a\mathbb{E}(X)$.

Proof: Let f be a simple function. It is trivially true in this case (Why?). We should be careful for the case where $\int f\, d\mu = \infty$ and $a = 0$. We see that $af = 0 \Rightarrow \int (af)\, d\mu = 0 = 0 \times \infty$ (By convention for extended Reals!) $= a \int f\, d\mu$, so the property holds.

Let f be a non-negative function. If $a = 0$, then the result is obvious. So, consider the case $a > 0$. It can be easily seen that $q \in \mathcal{S}(f) \Leftrightarrow aq \in \mathcal{S}(af)$.
$\int (af)\, d\mu = \sup\limits_{q' \in \mathcal{S}(af)} \int q'\, d\mu = \sup\limits_{aq \in \mathcal{S}(af)} \int (aq)\, d\mu = \sup\limits_{q \in \mathcal{S}(f)} \int (aq)\, d\mu = \sup\limits_{q \in \mathcal{S}(f)} a \int q\, d\mu = a \sup\limits_{q \in \mathcal{S}(f)} \int q\, d\mu = a \int f\, d\mu$.

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With the help of point (3) in section 17.2.2 from Lecture #17, proving the above properties for general measurable functions is not difficult, and is left as an exercise to the reader.

As mentioned in a previous lecture, we now prove the Inclusion-Exclusion property of probability measure using indicator random variables and their expectation.

Inclusion-Exclusion property of probability measures:

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $A_1, A_2, \cdots, A_n$ be elements of $\mathcal{F}$. Then,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}\left(A_i\right) - \sum_{i<j} \mathbb{P}\left(A_i \cap A_j\right) + \cdots + (-1)^{n-1} \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right).$$

Proof:

$$\begin{aligned}
\mathbb{I}_{\bigcup_{i=1}^{n} A_i} &= 1 - \mathbb{I}_{\bigcap_{i=1}^{n} A_i^c}, \\
&= 1 - \prod_{i=1}^{n} \mathbb{I}_{A_i^c}, \\
&= 1 - \prod_{i=1}^{n} (1 - \mathbb{I}_{A_i}). \\
&= \sum_{i=1}^{n} \mathbb{I}_{A_i} - \sum_{i<j} \mathbb{I}_{A_i}\mathbb{I}_{A_j} + \cdots + (-1)^{n-1}\Pi_{i=1}^{n}\mathbb{I}_{A_i}.
\end{aligned}$$

Taking expectation on both sides of the above equation yields the desired result, since $\mathbb{I}_{A_i}\mathbb{I}_{A_j} = \mathbb{I}_{A_i \cap A_j}$. ∎

Now, we summarize all the properties here:

[PAI 1]	$\int \mathbb{I}_A \, d\mu = \mu(A)$	$\mathbb{E}[\mathbb{I}_A] = \mathbb{P}(A)$
[PAI 2]	$g \geq 0, \Rightarrow \int g \, d\mu \geq 0$	$X \geq 0, \Rightarrow \mathbb{E}(X) \geq 0$
[PAI 3]	$g = 0 \, \mu\text{.a.e.}, \Rightarrow \int g \, d\mu = 0$	$X = 0 \, \text{a.s.}, \Rightarrow \mathbb{E}(X) = 0$
[PAI 4]	$\int (g+h) \, d\mu = \int g \, d\mu + \int h \, d\mu$	$\mathbb{E}(X+Y) = E(X) + E(Y)$
[PAI 5]	$0 \leq g \leq h \, \mu\text{.a.e.}, \Rightarrow \int g \, d\mu \leq \int h \, d\mu$	$0 \leq X \leq Y \, \text{a.s.}, \Rightarrow \mathbb{E}(X) \leq \mathbb{E}(Y)$
[PAI 6]	$g = h \, \mu\text{.a.e.}, \Rightarrow \int g \, d\mu = \int h \, d\mu$	$X = Y \, \text{a.s.}, \Rightarrow \mathbb{E}(X) = \mathbb{E}(Y)$
[PAI 7]	$g \geq 0 \, \mu\text{.a.e., and} \int g \, d\mu = 0, \Rightarrow g = 0 \, \text{a.e.}$	$X \geq 0 \, \text{a.s., and} \, \mathbb{E}(X) = 0, \Rightarrow X = 0 \, \text{a.s.}$.
[PAI 8]	$a \geq 0, \int (af) \, d\mu = a \int f d\mu$	$a \geq 0, \, \mathbb{E}(aX) = a\mathbb{E}(X)$

18.2 Exercise:

1. Show that if $g : \Omega \to [0, \infty]$ satisfies $\int g \, d\mu < \infty$, then $g < \infty$, μ.a.e..
2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $g : \Omega \to \mathbb{R}$ be a non-negative measurable function. Let λ be a lebesgue measure. Let f be a non-negative measurable function on the real line such that $\int f \, d\lambda = 1$. For any Borel set A, if $\mathbb{P}_1(A) = \int_A f \, d\lambda$, then prove that $\mathbb{P}_1$ is a probability measure.
3. Let X_1, X_2, ..., X_n be i.i.d. random variables for which $\mathbb{E}[X_1^{-1}]$ exists. Show that if $m \leq n$, then $\mathbb{E}[\frac{S_m}{S_n}] = \frac{m}{n}$, where $S_m = X_1 + X_2 + ... + X_m$.
4. Consider the Real line endowed with the Borel σ-algebra, and let $c \in \mathbb{R}$ be fixed. Then the *Dirac measure* at c, denoted as δ_c, is defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as follows. For any Borel set A, $\delta_c(A) = 1$ if $c \in A$, and $\delta_c(A) = 0$ if $c \notin A$. It is quite easy to see that $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \delta_c)$ is a measure space (indeed, it is a probability space). The Dirac measure is referred to as *unit impulse* in the engineering literature, and sometimes (incorrectly) called a Dirac delta "function".

(a) Let g be a non-negative, measurable function. Show that $\int g \; \mathrm{d}\delta_c = g(c)$.

Now, let us define a *counting measure* on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as $\mu(A) = \sum_{n=1}^{\infty} \delta_n(A)$. In words, $\mu(A)$simply counts the number of natural numbers contained in the Borel set A. In engineering parlance, the counting measure is called an impulse train.

(b) Let g be a non-negative, measurable function. Show that $\int g \; \mathrm{d}\delta_c = \sum_{n=1}^{\infty} g(n)$. Thus, summation is just a special case of integration. In particular, summation is nothing but integral with respect to the counting measure!