**EE5110: Probability Foundations for Electrical Engineers** **July-November 2015**

# Lecture 17: Integration and Expectation

*Lecturer: Dr. Krishna Jagannathan* *Scribe: Gopal Krishna Kamath M*

In this chapter, we introduce abstract integration, and in particular, define the integral of a measurable function, with respect to a measure. As a special case, the integral of a random variable with respect to a probability measure is known as the expectation of the random variable.

Our approach to defining the expectation of a random variable as an abstract integral serves to unify the definition. After all, you may recall from your undergraduate study of probability that the expectation of a random variable is defined via an integral if the random variable is continuous, and a summation if it is discrete. Of course, if the random variable were singular or a mixture, the elementary approach does not provide a simple definition of the expectation. On the other hand, the definition we are about to give is completely general; specifically, we do not have to provide separate definitions for different types of random variables.

In addition, the theory of abstract integration allows us to define the Lebesgue integral, which generalizes the notion of the Riemann integral from high-school calculus.

## 17.1 The Riemann Integral: A Review

Consider a function $f : \mathbb{R} \to \mathbb{R}$. Let $[a, b]$ be an interval in the domain of $f$, and $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be a partition of $[a, b]$. The lower and upper Riemann sums, denoted by $L_n$ and $U_n$ respectively, are defined as below:

$$\begin{aligned} L_n &\triangleq \sum_{i=1}^{n} \left( \inf_{x \in [x_i, x_{i+1}]} f(x) \right) \Delta x_i, \\ U_n &\triangleq \sum_{i=1}^{n} \left( \sup_{x \in [x_i, x_{i+1}]} f(x) \right) \Delta x_i. \end{aligned}$$

As $n$ increases in a manner such that each $\Delta x_i$ decreases to zero, it can be seen that $L_n$ is monotone increasing, while $U_n$ is monotone decreasing. So, as $n \to \infty$ it follows that $L_n$ and $U_n$ will both converge. The limits of $L_n$ and $U_n$ are called the lower and upper Riemann integrals, respectively. That is,

$$\begin{aligned} \underline{\int_a^b} f(x)\,\mathrm{d}x &\triangleq \lim_{n \to \infty} L_n, \\ \overline{\int_a^b} f(x)\,\mathrm{d}x &\triangleq \lim_{n \to \infty} U_n. \end{aligned}$$

It can be shown that the values of the Lower and Upper Riemann Integrals do not depend on the choice of

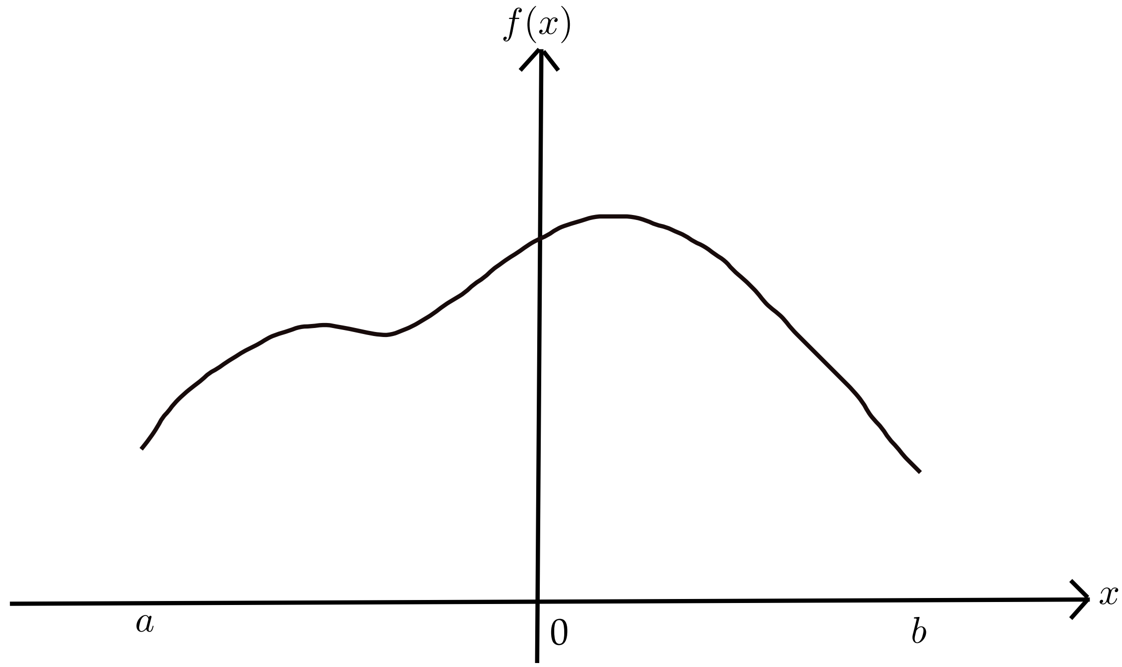


Figure 17.1: An arbitrary function $f$ over the interval $[a, b]$.

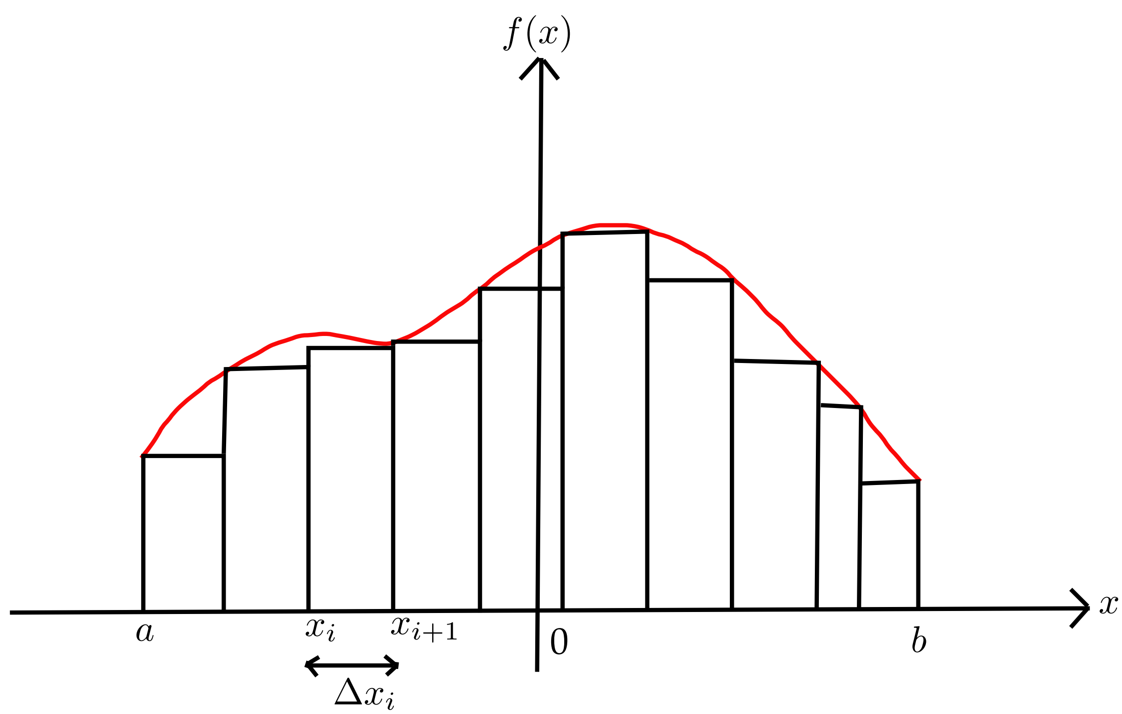


Figure 17.2: Lower Riemann approximation of $f$.

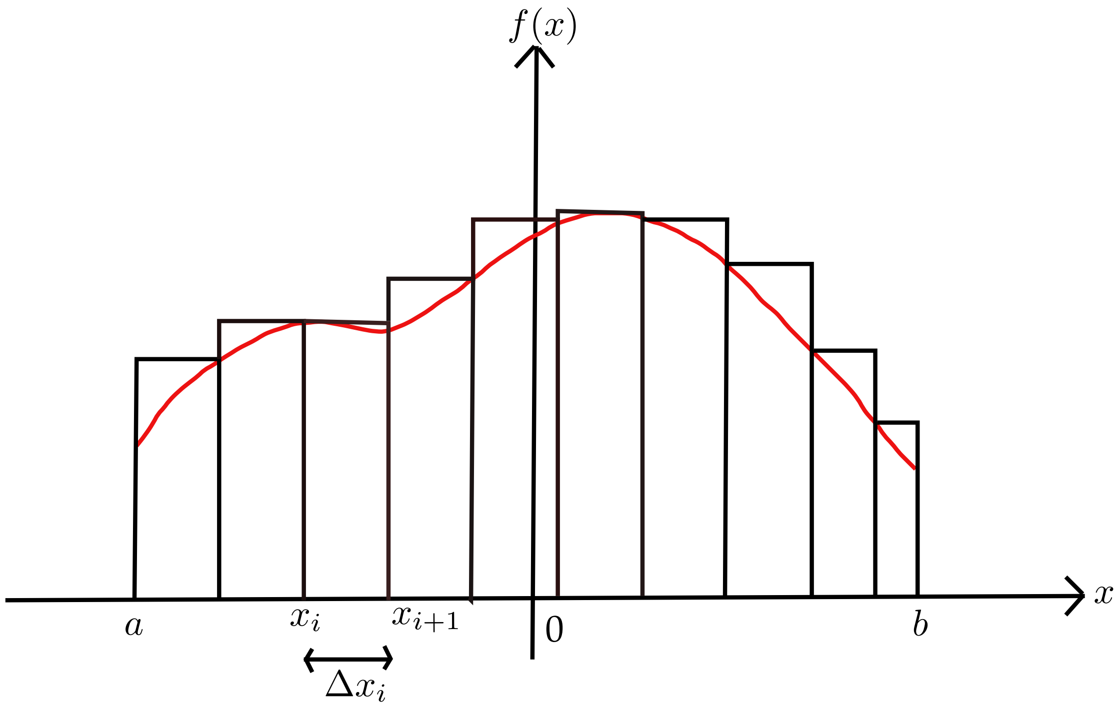


Figure 17.3: Upper Riemann approximation of $f$.

the partition. It is also clear that the following relation always holds

$$\underline{\int_a^b} f(x)\ \mathrm{d}x \leq \overline{\int_a^b} f(x)\ \mathrm{d}x.$$

**Definition 17.1** *A function $f$ is said to be Riemann Integrable if the values of the Lower Riemann Integral and the Upper Riemann Integral coincide. In such a case, the Riemann integral of $f$ is that common value. That is, the Riemann Integral of $f$ (when it exists), denoted by $\int_a^b f(x)\ dx$, is given by*

$$\int_a^b f(x)\ dx = \underline{\int_a^b} f(x)\ dx = \overline{\int_a^b} f(x)\ dx.$$

Figure (17.1) shows the graph of an arbitrary function $f$ over the interval $[a, b]$. Figures (17.2) and (17.3) show the lower and upper Riemann approximations of $f$ respectively, wherein $f$ is graphed in red for reference. The lower (resp. upper) Riemann sum is the area under the lower (resp. upper) Riemann approximation in figure (17.2) (resp. (17.3)). We can imagine the Lower (resp. Upper) Riemann Sum to be approximating the area under $f$ from below (resp. above). The intuition is that as the partitions become "finer", the Upper and Lower Riemann Sums converge to the area under $f$ from above and below respectively. Also notice that figures (17.2) and (17.3) also represent unequal partition sizes pictorially.

Next, we turn our attention to Abstract Integrals.

## 17.2 Abstract Integration

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $f : \Omega \to \mathbb{R}$ be a $\mathcal{F}$-measurable function. For any $A \in \mathcal{F}$, we would like to define

$$\int_A f \, \mathrm{d}\mu.$$

We will call the above quantity the integral of $f$ with respect to the measure $\mu$ over the $\mathcal{F}$-measurable set $A$. Also, in the interest of notational simplicity, we will use the following two notations interchangeably to mean the integral of the $\mathcal{F}$-measurable function $f$ with respect to the measure $\mu$ over the entire space

$$\int f \, \mathrm{d}\mu \equiv \int_\Omega f \, \mathrm{d}\mu.$$

Before we define the abstract integral, let us look at two very important special cases.

### 17.2.1 Special Cases

1. *The Lebesgue integral*: Let $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$
   Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel-measurable function, and in this case the integral

   $$\int f \, \mathrm{d}\lambda$$

   is called the *Lebesgue Integral* of $f$ over the Reals.
   The Lebesgue integral can be shown to be a generalisation of the Riemann integral. In particular, it allows us to integrate over arbitrary Borel sets, instead of just intervals. Moreover, we will see that the Lebesgue integral might exist even when the Riemann integral does not. However, if a function is Riemann integrable over an interval, then it is necessarily Lebesgue integrable, and the values of the two integrals will be equal.

2. *Expectation of a random variable:* Let $(\Omega, \mathcal{F}, \mu) = (\Omega, \mathcal{F}, \mathbb{P})$
   Let $X : \Omega \to \mathbb{R}$ be a random variable, and in this case the integral

   $$\int X \, \mathrm{d}\mathbb{P},$$

   is called the *Expectation* of the random variable $X$, and is denoted by $\mathbb{E}[X]$. Therefore,

   $$\mathbb{E}[X] \triangleq \int X \, \mathrm{d}\mathbb{P}.$$

Note that, so far, we have not *defined* what an abstract integral is. We have only introduced the notations and terminologies used. We now lay out the roadmap to defining the abstract integral.

### 17.2.2 Roadmap for defining the abstract integral

The abstract integral of an arbitrary, $\mathcal{F}$-measurable function $f$ is defined in four steps as outlined below:

1. First, we define the integral for *simple functions*, i.e., non-negative functions that take only finitely many values.

2. Second, we define the integral for non-negative functions. This is done by approximating the function by simple functions, thus allowing us to define the integral of the non-negative function in terms of the integrals of the simple functions.

3. Third, we write the arbitrary function $f$ as $f = f_+$ - $f_-$, where $f_+$ and $f_-$ are non-negative functions which correspond to the positive and negative components of $f$. Then, we define the integral of $f$ as
$$\int f \, \mathrm{d}\mu = \int f_+ \, \mathrm{d}\mu - \int f_- \, \mathrm{d}\mu.$$

4. And last, we define
$$\int_A f \, \mathrm{d}\mu \triangleq \int f \mathbb{I}_A \, \mathrm{d}\mu.$$

### 17.2.3 Abstract Integrals of Simple Functions

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f : \Omega \to \mathbb{R}$ be a $\mathcal{F}$-measurable function.

**Definition 17.2** *A function $f$ is said to be a* simple *function if it can be written as*

$$f(\omega) = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i}(\omega), \ \forall \ \omega \in \Omega, \tag{17.1}$$

*where $a_i \geq 0 \ \forall \ i \in \{1, 2, \ldots, n\}$, and $A_i \in \mathcal{F} \ \forall \ i \in \{1, 2, \ldots, n\}$.*

**Remark: 17.3** *Note that $f(\omega)$ written in this form is not unique. This problem is circumvented using the "canonical" representation, wherein we restrict the $a_i$'s to be distinct and the $A_i$'s to be disjoint. It can be verified that this restriction enforces uniqueness of representation. Henceforth, whenever the term "simple function" is used, it will be taken implicitly to be in the canonical form.*

Figure (17.4) shows the canonical representation of a simple random variable $X$ taking 4 values such that $\omega \in A_i \implies X(\omega) = a_i \ \forall \ i \in \{1, 2, 3, 4\}$. As the figure shows, disjoint events are mapped to distinct, non-negative real numbers.

**Definition 17.4** *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f \geq 0$ be a simple function with the canonical representation* (17.2). *The abstract integral of $f$ with respect to the measure $\mu$ is defined as*

$$\int f \ d\mu \triangleq \sum_{i=1}^{n} a_i \mu(A_i).$$

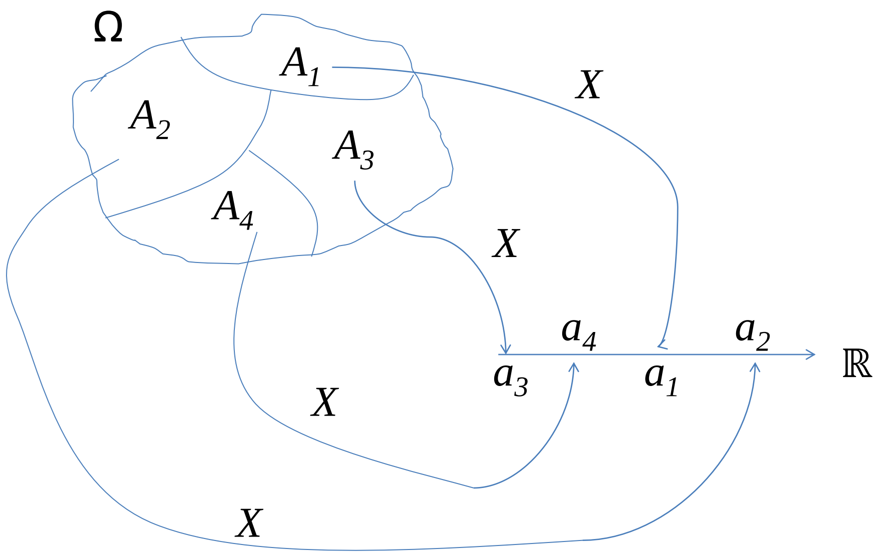


Figure 17.4: Canonical representation of a Simple Random Variable taking 4 values.

**Example 1:-** Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, and define $f(\omega) = u(\omega) + u(\omega - 1)$ - $2u(\omega - 3)$, where $u(.)$ is the Heavyside Step function. The canonical representation of this simple function is $f(\omega) = \mathbb{I}_{[0,1]} + 2\mathbb{I}_{[1,3]}$. Therefore, the Lebesgue integral of this function is

$$\begin{aligned}
\int f \text{ d}\lambda &= 1 \times \lambda([0,1]) + 2 \times \lambda([1,3]), \\
&= 1 \times 1 + 2 \times 2, \\
&= 5.
\end{aligned}$$

We note that the value of the integral equals the area under the curve.

**Example 2:-** Consider the probability space $(\Omega = \{H, T\}^n, \mathcal{F}, \mathbb{P})$ and let $X : \Omega \to \mathbb{R}$ be a simple random variable such that $\mathbb{P}(\{H\}) = p$. This can be considered a model for $n$ independent coin tosses. If $X(\omega)$ represents the number of heads, then the expected value of $X$ (i.e. the integral of $X$ with respect to $\mathbb{P}$) can be calculated as

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{i=1}^{n} i\mathbb{P}(X = i), \\
&= \sum_{i=1}^{n} i \binom{n}{i} p^i (1-p)^{n-i}, \\
&= np.
\end{aligned}$$

**Example 3:-** Consider the Dirichlet function ($D$) defined to take on the value 1 on the rationals in the interval $[0, 1]$ and the value 0 on the irrationals in the interval $[0, 1]$. For this function,

$$\begin{aligned} \underline{\int_0^1} D(x)\ \mathrm{d}x &= 0, \\ \overline{\int_0^1} D(x)\ \mathrm{d}x &= 1. \end{aligned}$$

Hence, the Dirichlet function is not Riemann Integrable. On the other hand, the Dirichlet function is a simple function with the canonical representation

$$D(\omega) = \mathbb{I}_{\mathbb{Q}\cap[0,1]}.$$

Therefore, the Lebesgue Integral of the Dirichlet function is given by

$$\begin{aligned} \int D(\omega)\ \mathrm{d}\lambda &= 1 \times \lambda(\mathbb{Q} \cap [0, 1]), \\ &= 0. \end{aligned}$$

This is because every partition of the horizontal axis, no matter how fine, contains both rational and irrational points. Therefore, we see that the Dirichlet function is trivially Lebesgue Integrable while not being Riemann Integrable.

### 17.2.4 Abstract Integral of non-negative functions

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $f : \Omega \to \mathbb{R}_+$ be a non-negative, $\mathcal{F}$-measurable function. Denote by $S(f)$ the collection of all simple functions $q : \Omega \to \mathbb{R}_+$ such that $q(\omega) \leq f(\omega)\ \forall\ \omega \in \Omega$. That is, given a non-negative function $f$, we collect all the simple functions $q$'s that approximate $f$ from below. Having done this, we now define the abstract integral of $f$ as follows:

**Definition 17.5** *The abstract integral of $f$ with respect to the measure $\mu$ is defined as*

$$\int f\ d\mu \triangleq \sup_{q\in S(f)} \int q\ d\mu. \tag{17.2}$$

Since $q$'s are simple functions, calculation of their integral is known. The above equation gives a way to find the integral of any non-negative function. While being mathematically well-defined, (17.2) does not directly yield a practical method to compute the integral. We will address this issue later.

### 17.2.5 Abstract Integral of arbitrary functions

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $f : \Omega \to \mathbb{R}$ be any arbitrary $\mathcal{F}$-measurable function. Then, in order to evaluate $\int f\ \mathrm{d}\mu$, we first write $f$ as $f = f_+$ - $f_-$, where $f_+ \triangleq \max(f, 0) \geq 0$ and $f_- \triangleq -\min(f, 0) \geq 0$. We then define the integral of $f$ with respect to $\mu$ as

$$\int f\ \mathrm{d}\mu \triangleq \int f_+\ \mathrm{d}\mu - \int f_-\ \mathrm{d}\mu, \tag{17.3}$$

wherein the integrals of $f_+$ and $f_-$ as calculated as in the previous section. Since $f_+$ and $f_-$ are nonnegative functions, both integrals on right hand side of (17.3) is well defined. The above definition is meaningful, as long as at least one of the integrals on the right hand side of (17.3) is finite. The integral of $f$ is left undefined if the integrals of $f_+$ and $f_-$ are both infinite.

### 17.2.6 Abstract Integral of arbitrary functions over a given set

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $f : \Omega \to \mathbb{R}$ be any arbitrary $\mathcal{F}$-measurable function, and $A \in \mathcal{F}$. Define $g \triangleq f\mathbb{I}_A$. That is, we consider the function $f$ restricted to the set $A$. This is an $\mathcal{F}$-measurable function since it is a product of two $\mathcal{F}$-measurable functions. Its integral can be calculated as mentioned in the previous section. Therefore,

$$\int_A f \, \mathrm{d}\mu = \int f\mathbb{I}_A \, \mathrm{d}\mu = \int g \, \mathrm{d}\mu = \int g_+ \, \mathrm{d}\mu - \int g_- \, \mathrm{d}\mu.$$

## 17.3 Exercises

1. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, and define $f : \mathbb{R} \to \mathbb{R}$. Find out the Lebesgue integral of the function $f$ for the following cases,

   (a)
   $$f(\omega) = \begin{cases} \omega, & \text{for } \omega = 0, 1, .., n \\ 0, & \textit{elsewhere.} \end{cases}$$

   (b)
   $$f(\omega) = \begin{cases} 1, & \text{for } \omega = \mathbb{Q}^c \cap [0, 1] \\ 0, & \textit{elsewhere.} \end{cases}$$

   (c)
   $$f(\omega) = \begin{cases} n, & \text{for } \omega = \mathbb{Q}^c \cap [0, n] \\ 0, & \textit{elsewhere.} \end{cases}$$

2. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and define the random variable $X : \Omega \to \mathbb{R}$. Find $\mathbb{E}[X]$ for the following cases,

   (a) $\Omega = \{\omega_1, \omega_2, .., \omega_n\}$, with $\mathbb{P}(\omega_i) = 1/n$ for $i = 1, 2, .., n$ and $X = \mathbb{I}_A$, where $A = \{\omega_1, \omega_2, .., \omega_m\}$ with $1 \leq m \leq n$.

   (b) In part (a) if X is defined as follows,
   $$X(\omega_i) = \begin{cases} i, & \text{for } \omega_i \in A \\ 0, & \textit{elsewhere.} \end{cases}$$