Lecture 12: Multiple Random Variables and Independence

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12.1   Multiple Random Variables

In this lecture, we consider multiple random variables defined on the same probability space. To begin with, let us consider two random variables X and Y, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is important to understand that the realizations of X and Y are governed by the same underlying randomness, namely $\omega \in \Omega$. For example, the underlying sample space could be something as complex as the weather on a particular day; the random variable X could denote the temperature on that day, and another random variable Y, the humidity level. Since the same underlying outcome governs both X and Y, it is reasonable to expect X and Y to posses a certain degree of interdependence. In the above example, a high temperature on a given day usually says something about the humidity.

In Figure (12.1), the top picture shows two random variables X and Y, each mapping Ω to $\mathbb{R}$. These two random variables are measurable functions from the same probability space to the real line. The bottom picture in Figure (12.1) shows $(X(\cdot), Y(\cdot))$ mapping Ω to $\mathbb{R}^2$. Indeed, the bottom picture is more meaningful, since it captures the interdependence between X and Y.

Now, an important question arises: is the function $(X(\cdot), Y(\cdot)) : \Omega \to \mathbb{R}^2$ measurable, given that X and Y are measurable functions? In order to pose this question properly and answer it, we first need to define the Borel σ-algebra on $\mathbb{R}^2$. The Borel σ-algebra on $\mathbb{R}^2$ is the σ-algebra generated by the class $\pi\left(\mathbb{R}^2\right) \triangleq \{(-\infty, x] \times (-\infty, y] \,|\, x, y \in \mathbb{R}\}$. That is,

$$\mathcal{B}(\mathbb{R}^2) = \sigma\left(\pi\left(\mathbb{R}^2\right)\right).$$

The following theorem asserts that whenever X and Y are random variables, the function $(X, Y) : \Omega \to \mathbb{R}^2$ is $\mathcal{F}$-measurable, in the sense that the pre-images of Borel sets on $\mathbb{R}^2$ are necessarily events.

Theorem 12.1 *Let X and Y be two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Then, $(X(\cdot), Y(\cdot)) : \Omega \to \mathbb{R}^2$ is $\mathcal{F}$-measurable, i.e., the pre-images of Borel sets on $\mathbb{R}^2$ under $(X(\cdot), Y(\cdot))$ are events.*

Proof: Let $\mathcal{G}$ be the collection of all subsets of $\mathbb{R}^2$ whose pre-images under $(X(\cdot), Y(\cdot))$ are events. To prove the theorem, it is enough to prove that $\mathcal{B}\left(\mathbb{R}^2\right) \subseteq \mathcal{G}$.

Claim 1: $\mathcal{G}$ is a σ- algebra of subsets of $\mathbb{R}^2$.

Next, note that $\{\omega | X(\omega) \leq x\}$, $\{\omega | Y(\omega) \leq y\} \in \mathcal{F}$, $\forall x, y \in \mathbb{R}$, since X and Y are random variables. Thus, $\{\omega | X(\omega) \leq x\} \cap \{\omega | Y(\omega) \leq y\} \in \mathcal{F}$, $\forall x, y \in \mathbb{R}$, since $\mathcal{F}$ is a σ- algebra.

So, $\{\omega | X(\omega) \leq x, Y(\omega) \leq y\} \in \mathcal{F}$, $\forall x, y \in \mathbb{R} \Rightarrow (-\infty, x] \times (-\infty, y] \in \mathcal{G}$, $\forall x, y \in \mathbb{R}$ (from the definition of $\mathcal{G}$) $\Rightarrow \pi\left(\mathbb{R}^2\right) \subseteq \mathcal{G} \Rightarrow \sigma\left(\pi\left(\mathbb{R}^2\right)\right) \subseteq \sigma\left(\mathcal{G}\right) \Rightarrow \mathcal{B}\left(\mathbb{R}^2\right) \subseteq \mathcal{G}.$ ∎

Since the pre-images of Borel sets on $\mathbb{R}^2$ are events, we can assign probabilities to them. This leads us to the definition of the joint probability law.
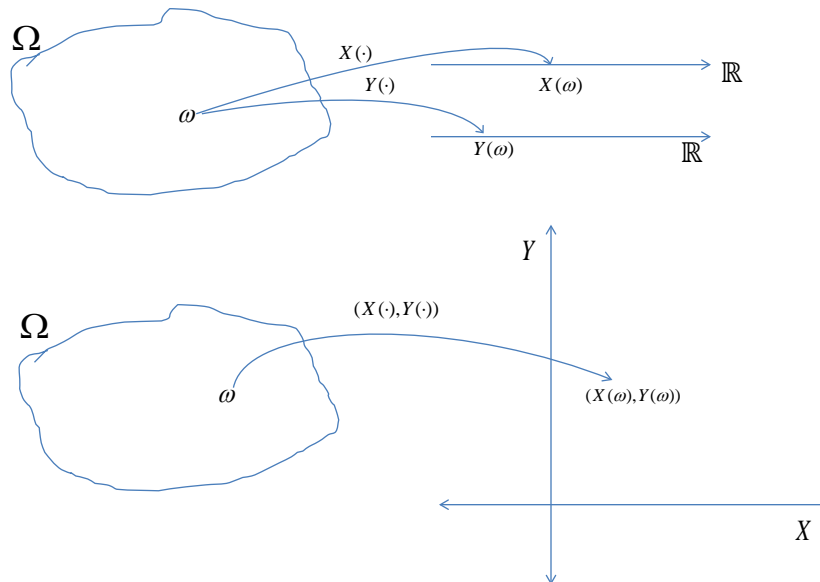
Figure 12.1: Illustration of Multiple Random Variables

Definition 12.2 *The joint probability law of the random variables X and Y is defined as:*

$$\mathbb{P}_{X,Y}(B) = \mathbb{P}\left(\{\omega \in \Omega \,|\, (X(\omega), Y(\omega)) \in B\}\right), \quad B \in \mathcal{B}(\mathbb{R}^2),$$

where $\mathcal{B}(\mathbb{R}^2)$ is the Borel σ-algebra on $\mathbb{R}^2$.

In particular, when $B = (-\infty, x] \times (-\infty, y]$, we have

$$\mathbb{P}_{X,Y}\left((-\infty, x] \times (-\infty, y]\right) = \mathbb{P}\left(\{\omega \,|\, X(\omega) \le x, Y(\omega) \le y\}\right). \tag{12.1}$$

The LHS in (12.1) is well defined, and hence the RHS in (12.1) is well defined and is called as the joint CDF of X and Y.

12.2   Joint CDF

Definition 12.3 *Let X and Y be two random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The joint CDF of X and Y is defined as follows:*

$$F_{X,Y}(x,y) = \mathbb{P}\left(\{\omega \,|\, X(\omega) \le x, Y(\omega) \le y\}\right), \quad \forall x, y \in \mathbb{R}.$$

In short hand, we write $F_{X,Y}(x,y) = \mathbb{P}\left(X \le x, Y \le y\right).$

12.2.1 Properties of joint CDF:

1. $\lim\limits_{\substack{x\to\infty \\ y\to\infty}} F_{X,Y}(x,y) = 1, \quad \lim\limits_{\substack{x\to-\infty \\ y\to-\infty}} F_{X,Y}(x,y) = 0.$

   Proof: Let $\{x_n\}$ and $\{y_n\}$ be two unbounded, monotone-increasing sequences. We have

   $$
   \begin{aligned}
   \lim_{\substack{x\to\infty \\ y\to\infty}} F_{X,Y}(x,y) &= \lim_{\substack{x\to\infty \\ y\to\infty}} \mathbb{P}(X \le x, Y \le y), \\
   &= \lim_{n\to\infty} \mathbb{P}(X \le x_n, Y \le y_n), \\
   &\overset{(a)}{=} \mathbb{P}\left(\bigcup_{n=1}^{\infty} \{\omega : X(\omega) \le x_n, Y(\omega) \le y_n\} \right), \\
   &= \mathbb{P}(\Omega), \\
   &= 1,
   \end{aligned}
   $$

   where (a) is due to continuity of probability measures (Lecture #5, Property 6). Proof of the other part follows on the similar lines and is left as an exercise to the reader.
   Note that the order of the two limits does not matter here. ∎

2. *Monotonicity:* For any $x_1 \le x_2, y_1 \le y_2, F_{X,Y}(x_1,y_1) \le F_{X,Y}(x_2,y_2)$.
   Proof: Let $x_1 \le x_2$ and $y_1 \le y_2$. Clearly, events $\{X \le x_1, Y \le y_1\} \subseteq \{X \le x_2, Y \le y_2\}$. Then, $\mathbb{P}(X \le x_1, Y \le y_1) \le \mathbb{P}(X \le x_2, Y \le y_2) \Rightarrow F_{X,Y}(x_1,y_1) \le F_{X,Y}(x_2,y_2)$. ∎

3. $F_{X,Y}$ is continuous from above, i.e., $\lim\limits_{\substack{u\to0^+ \\ v\to0^+}} F_{X,Y}(x+u, y+v) = F_{X,Y}(x,y), \ \forall x, y \in \mathbb{R}$.
   Exercise: Prove this.

4. $\lim\limits_{y\to\infty} F_{X,Y}(x,y) = F_X(x)$
   Proof: Let $\{y_n\}$ be an unbounded, monotone-increasing sequence. Then, $\lim\limits_{y\to\infty} F_{X,Y}(x,y) = \lim\limits_{n\to\infty} F_{X,Y}(x,y_n)$.
   Hence,

   $$
   \begin{aligned}
   \lim_{n\to\infty} F_{X,Y}(x,y_n) &= \lim_{n\to\infty} \mathbb{P}(X \le x, y \le y_n), \\
   &\overset{(a)}{=} \mathbb{P}\left(\bigcup_{n=1}^{\infty} \{\omega : X(\omega) \le x, Y(\omega) \le y_n\} \right), \\
   &= \mathbb{P}(\omega : X(\omega) \le x), \\
   &= F_X(x),
   \end{aligned}
   $$

   where (a) is due to continuity of probability measure.(Lecture #5, Property 6). ∎
   Using the above property, we can calculate the marginal CDFs from joint CDF. However, the joint CDF cannot be obtained from the marginals alone, since the marginals do not capture the inter-dependence of X and Y.

12.3 The σ-algebra generated by a random variable

Before we proceed to define the independence of random variables, it is useful to understand the notion of the σ-*algebra generated by a random variable*. We first state an elementary result that holds for any arbitrary function.

Proposition 12.4 *Let Ω and S be two non-empty sets and let $f : \Omega \longrightarrow S$ be a function. If $\mathcal{H}$ is a σ-algebra of subsets of S, then $\mathcal{G} \triangleq \{A \mid A = f^{-1}(B)\, , B \in \mathcal{H}\}$ is a σ-algebra of subsets of Ω.*

In words, Proposition (12.4) says that the collection of pre-images of all the sets belonging to some σ-algebra on the range of a function, is a σ-algebra on the domain of that function.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability Space and $X : \Omega \longrightarrow \mathbb{R}$ be a random variable. X in turn induces the probability triple $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_X)$ on the real line.

Definition 12.5 *The σ-algebra generated by the random variable X is defined as*

$$\sigma(X) \triangleq \{E \subseteq \Omega | E = X^{-1}(B),\ \forall B \in \mathcal{B}(\mathbb{R})\}. \tag{12.2}$$

Proposition (12.4) asserts that $\sigma(X)$ defined above is indeed a σ-algebra on Ω.

Proposition 12.6 $\sigma(X) \subseteq \mathcal{F}$, *i.e., the σ-algebra generated by X is a sub-σ-algebra of $\mathcal{F}$.*

Figure (12.3) shows a pictorial representation of the σ-algebra generated by X. Each Borel set B maps back to an event E. A collection of all such preimages of Borel sets constitutes the σ-algebra generated by X. Thus, $\sigma(X)$ is a σ-algebra that consists precisely of those events whose occurrence or otherwise is completely determined by looking at the realised value $X(\omega)$. To get a more concrete idea of this concept, let us look at the following examples:
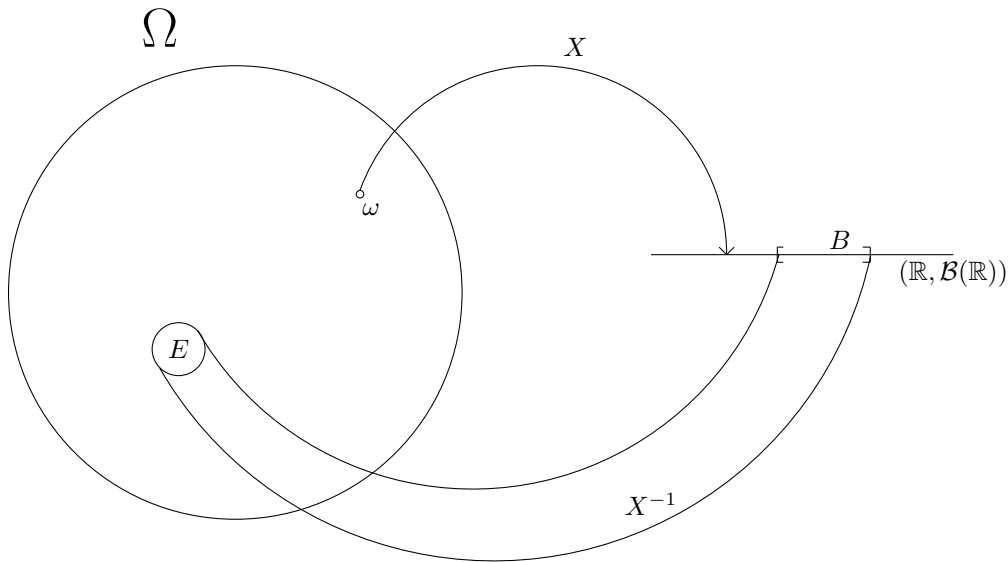

Figure 12.2: The collection of the pre-images of all Borel Sets is the σ-algebra generate by the random variable X, denoted $\sigma(X)$.

Example 1:- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A \in \mathcal{F}$ be some event. Consider the Indicator random variable of event A, $\mathbb{I}_A$. It is easy to see that $\sigma(\mathbb{I}_A) = \{\emptyset, A, A^C, \Omega\}$. Also, $\sigma(\mathbb{I}_A) \subset \mathcal{F}$.

Example 2:- Let $([0,1], \mathcal{B}([0,1]), \lambda)$ be the probability space in consideration, and consider a random variable $X(\omega) = \omega, \forall \omega \in \Omega$. It can be seen that $\sigma(X) = \mathcal{F}$.

Remark: 12.7 *As seen from the above two examples, $\sigma(X)$ could either be "small" (as seen in example 1 above) or as "large" as the σ-algebra $\mathcal{F}$ itself (as seen in example 2 above).*

Now, we introduce the important notion of independence of random variables.

12.4 Independence of Random Variables

Definition 12.8 *Random variables X and Y are said to be independent if $\sigma(X)$ and $\sigma(Y)$ are independent σ-algebras.*

In other words, X and Y are independent if, for any two borel sets B_1 and B_2 on $\mathbb{R}$, the events $\{\omega : X(\omega) \in B_1\}$ and $\{\omega : Y(\omega) \in B_2\}$ are independent i.e.,

$$\mathbb{P}\left(\{\omega : X(\omega) \in B_1\} \bigcap \{\omega : Y(\omega) \in B_2\}\right) = \mathbb{P}\left(\{\omega : X(\omega) \in B_1\}\right) \mathbb{P}\left(\{\omega : Y(\omega) \in B_2\}\right), \forall B_1, B_2 \in \mathcal{B}(\mathbb{R}).$$

The following theorem gives a useful characterization of independence of random variables, in terms of the *joint CDF being equal to the product of the marginals*.

Theorem 12.9 *X and Y are independent if and only if*

$$F_{X,Y}(x,y) = F_X(x)F_Y(y).$$

Proof: First, we prove the necessary part, which is straightforward. Let X and Y are independent. Consider $B_1 \in \mathcal{B}(\mathbb{R})$, $B_2 \in \mathcal{B}(\mathbb{R})$ then the events $\{\omega | X(\omega) \in B_1\}$ and $\{\omega | Y(\omega) \in B_2\}$ are independent (due to definition (12.8)) $\Rightarrow \mathbb{P}(X \in B_1, Y \in B_2) = \mathbb{P}(X \in B_1)\mathbb{P}(Y \in B_2) \Rightarrow \mathbb{P}_{X,Y}(B_1 \times B_2) = \mathbb{P}(X \in B_1)\mathbb{P}(Y \in B_2)$. But, this is true for all borel sets in $\mathbb{R}$. In particular, choose $B_1 = (-\infty, x]$ and $B_2 = (-\infty, y]$ then we get $F_{X,Y}(x,y) = F_X(x)F_Y(y), \forall x, y \in \mathbb{R}$ which completes the proof of the necessary part.

The sufficiency part is more involved; refer [1][Section 4.2]. ∎

Definition 12.10 *$X_1, X_2, \cdots X_n$ random variables are said to be independent if σ-algebras $\sigma(X_1), \sigma(X_2), \cdots, \sigma(X_n)$ are independent i.e., for any $B_i \in \mathcal{B}(\mathbb{R}), 1 \le i \le n$, we have*

$$\mathbb{P}(X_1 \in B_1, X_2 \in B_2, \cdots X_n \in B_n) = \prod_{i=1}^{n} \mathbb{P}(X_i \in B_i).$$

Theorem 12.11 *$X_1, X_2, \cdots X_n$ are independent if and only if*

$$F_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots x_n) = \prod_{i=1}^{n} F_{X_i}(x_i).$$

Proof: Refer [1][Section 4.2]. ∎

Finally, we define independence for an arbitrary family of random variables.

Definition 12.12 *An arbitrary family of random variables, $\{X_i, \in I\}$, is said to be independent if the $\sigma-$algebras $\{\sigma(X_i), i \in I\}$ are independent (Lecture #9, Section 9.2, Definition 9.7).*

12.5   Exercises

1. Prove **Claim 1** under Theorem 12.1.

2. For random variables X and Y defined on same probability space, with joint CDF $F_{X,Y}(x,y)$, prove that $\lim\limits_{\substack{x \to -\infty \\ y \to -\infty}} F_{X,Y}(x,y) = 0$.

3. Prove Propositions (12.4) and (12.6).

4. *[Quiz II 2014]* Suppose X and Y are independent random variables, and $f(X)$ and $g(Y)$ are functions of X and Y respectively. Will the random variables $f(X)$ and $g(Y)$ be independent? Justify your answer.

References

[1]   DAVID WILLIAMS, "Probability with Martingales", *Cambridge University Press*, Fourteenth Printing, 2011.