EE5110: Probability Foundations for Electrical Engineers **July-November 2015**

Lecture 10: The Borel-Cantelli Lemmas

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The Borel-Cantelli lemmas are a set of results that establish if certain events occur infinitely often or only finitely often. We present here the two most well-known versions of the Borel-Cantelli lemmas.

Lemma 10.1 (First Borel-Cantelli lemma) *Let $\{A_n\}$ be a sequence of events such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. Then, almost surely, only finitely many $A_n{}'s$ will occur.*

Lemma 10.2 (Second Borel-Cantelli lemma) *Let $\{A_n\}$ be a sequence of independent events such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$. Then, almost surely, infinitely many $A_n{}'s$ will occur.*

It should be noted that only the second lemma stipulates independence. The event "A_n occurs infinitely often (A_n *i.o.*)" is the set of all $\omega \in \Omega$ that belong to infinitely many $A_n{}'s$. It is defined as

$$\{A_n \ i.o.\} \triangleq \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{m=n}^{\infty} A_m}_{B_n}. \tag{10.1}$$

Here, B_n is the event that atleast one of $A_n, A_{n+1}, A_{n+2}, \ldots$ occur. Hence, $\{A_n \ i.o.\}$ is the event that for every $n \in \mathbb{N}$, there exists atleast one $m \in \{n, n+1, \ldots, \infty\}$ such that A_m occurs. Taking complement of both sides in (10.1), we get the expression for the event that A_n occurs finitely often (A_n *f.o.*)

$$\{A_n \ f.o.\} = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m{}^c.$$

In order to prove the Borel-Cantelli lemmas, we require the following lemma.

Lemma 10.3 *If $\sum_{i=1}^{\infty} p_i = \infty$, then $\lim_{n\to\infty} \prod_{i=1}^{n} (1-p_i) = 0$.*

Proof: Since $\ln(1-p_i) \leq -p_i$,

$$\begin{aligned}
\prod_{i=1}^{n} (1-p_i) &= \prod_{i=1}^{n} e^{\ln(1-p_i)} \\
&\leq \prod_{i=1}^{n} e^{-p_i} \\
&= e^{-\sum_{i=1}^{n} p_i}.
\end{aligned}$$

Taking limit on both the sides gives

$$\begin{aligned}
\lim_{n\to\infty} \prod_{i=1}^{n} (1-p_i) &\leq \lim_{n\to\infty} e^{-\sum_{i=1}^{n} p_i} \\
&= 0.
\end{aligned}$$

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We now proceed towards proving the Borel-Cantelli lemmas.

Proof:

1. First, note that the assumption $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ implies $\lim_{n\to\infty} \sum_{m=n}^{\infty} \mathbb{P}(A_m) = 0$. Next, since $B_{n+1} \subset B_n$, we can use continuity of probability to write

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right) &= \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right) \\
&= \lim_{n\to\infty} \mathbb{P}(B_n) \\
&= \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{m=n}^{\infty} A_m\right) \\
&\le \lim_{n\to\infty} \sum_{m=n}^{\infty} \mathbb{P}(A_m) \\
&= 0.
\end{aligned}$$

   We have used the union bound in writing the '$\le$' above. Since $\mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i\right) \ge 0$, we conclude that $\mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i\right) = 0$. This implies that, A_n occurs finitely often with probability 1.

2. The event that A_n occurs finitely often (A_n $f.o.$) is given by

$$\{A_n \ f.o.\} = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} {A_i}^c.$$

   Now,

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i^c\right) &\le \sum_{n=1}^{\infty} \mathbb{P}\left(\bigcap_{i=n}^{\infty} A_i^c\right) \qquad \text{(Using union bound)} \\
&= \sum_{n=1}^{\infty} \lim_{m\to\infty} \mathbb{P}\left(\bigcap_{i=n}^{m} A_i^c\right) \qquad \text{(By continuity of probability)} \\
&= \sum_{n=1}^{\infty} \prod_{i=n}^{m} \mathbb{P}(A_i^c) \qquad \text{(By independence)} \\
&= 0 \qquad \text{(By lemma 10.3)}
\end{aligned} \tag{10.2}$$

   Since $\mathbb{P}\left(\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i^c\right) \ge 0$, we conclude that $\mathbb{P}\left(\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i^c\right) = 0$. This implies that, A_n occurs infinitely often with probability 1.

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We now illustrate the usefulness of the Borel-Cantelli lemmas using an example. Consider an experiment in which a coin is tossed independently many times. Let $\mathbb{P}(H_n)$ be the probability of obtaining head at the n^{th} toss (and similarly for T_n).

1. Suppose $\mathbb{P}(H_n) = \frac{1}{n}$, $n \geq 1$. Then $\sum_{n=1}^{\infty} \mathbb{P}(H_n) = \infty$. By the second Borel-Cantelli lemma, it follows that almost surely, infinitely many heads will occur. This might appear surprising at first sight, since as n becomes large, the probability of getting heads becomes vanishingly small. However, the decay rate $1/n$ is not 'fast enough.' In particular, for any n we choose (no matter how large), there occurs a head beyond n almost surely!

2. Suppose now that $\mathbb{P}(H_n) = \frac{1}{n^2}$. Then $\sum_{n=1}^{\infty} \mathbb{P}(H_n) < \infty$, and hence by the first Borel-Cantelli lemma, almost surely, only finitely many heads will occur. In this case, the occurence of heads is decreasing fast enough that after a finite n, there will almost surely be no heads. Note that independence is not required in this case.

Exercises

1. Suppose that a monkey sits in front of a computer and starts hammering keys randomly on the keyboard. Show that the famous Shakespeare monologue starting All the worlds a stage will eventually appear (with probability 1) in its entirety on the screen, although our monkey is not particularly known for its good taste in literature. You can make reason- able additional assumptions to form a probability model; for example, you can assume that the monkey picks characters uniformly at random on the keyboard, and that the successive key strokes are independent.

2. *[MIT OCW problem set]* Let $A_n, n \geq 1$ be a sequence of events such that $\mathbb{P}(A_n) \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} \mathbb{P}\left(A_n^c \cap A_{n+1}\right) < \infty$$

Show that almost surely, only finitely many of the A_ns will occur.

3. **Online dating**: On a certain day, Alice decides that she will start looking for a potential life partner on an online dating portal. She decides that everyday, she will pick a guy uniformly at random from among the male members of the dating portal, and go out on a date with him. What Alice does not know, is that her neighbor Bob is interested in dating her. Being of a shy disposition, Bob decides that he will not ask Alice out himself. Instead, he decides that he will go out on a date with Alice only on the days that Alice happens to pick him from the dating portal, of which he is already a member. For the first two parts, assume that 50 new male members and 40 new female members join the dating portal everyday.

(a) What is the probability that Alice and Bob would have a date on the nth day? Do you think Bob and Alice would eventually stop meeting? Justify your answer, clearly stating any additional assumptions.

(b) Now suppose that Bob also picks a girl uniformly at random everyday, from among the female members of the portal, and that Alice behaves exactly as before. Assume also that Bob and Alice will meet on a given day if and only if they both happen to pick each other. In this case, do you think Bob and Alice would eventually stop meeting?

(c) For this part, suppose that Alice and Bob behave as in part (a), i.e.,Alice picks a guy uniformly at random, but Bob is only interested in dating Alice. However, the number of male members in the portal increases by 1 percent everyday. Do you think Bob and Alice would eventually stop meeting?

4. Let $\{S_n : n \geq 0\}$ be a simple random walk which moves to the right with probability p at each step, and suppose that $S_0 = 0$. Write $X_n = S_n - S_{n-1}$.

(a) Show that $\{S_n = 0 \; i.o\}$ is not a tail event of the sequence $\{X_n\}$.

(b) Show that $\mathbb{P}\left(S_n = 0 \; i.o\right) = 0$ if $p \neq \frac{1}{2}$