Advanced Topics in VLSI EE-6361

Probability Tutorial

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1. Suppose five fair coins are tossed. Let E be the event that three coins show heads. Define the indicator random variable I_E to be 1 if E occurs and 0 otherwise. For what outcomes in the original sample space does I_E equal to 1? What is the value of $P(\{I_E = 1\})$. What is the mean and variance of I_E?

$$I_E = \begin{cases} 1 & \text{if E occurs} \\ 0 & \text{otherwise} \end{cases}$$

2. Let I_α be an indicator RV that indicates that a value drawn from a Standard Normal distribution is $\geq \alpha$. Calculate $E[I_\alpha]$ for (a) $\alpha = 0$ (b) $\alpha = 0.75$ and (c) $\alpha = -0.75$

3. A geometric RV, X, is an integer valued RV (≥ 1) and is defined to be the number flips of a coin required to get a head. Assume that the coin shows a head with probability p. Find the probability mass function $P(\{X = n\})$ for $n \geq 1$.

4. A binomial RV, X, is an integer valued RV(≥ 0) that represents the number of successes of a Bernoulli RV in n trials. If p represents the probability of success of the Bernoulli RV, find the following

    - Probability mass function $P(\{X = k\})$ for $k \geq 0$,
    - Mean of X
    - Variance of X

5. A coin having probability p of coming up heads is successively flipped until the r^{th} head appears. Show that the random variable X, number of flips required, will be $n \geq r$, is defined by the probability mass function

$$P(\{X = n\}) = \begin{pmatrix} n-1 \\ r-1 \end{pmatrix} p^r (1-p)^{n-r} \ldots n \geq r$$

6. Consider a memory with N cells that has a row redundancy fix of up to K rows. Assume that each cell can fail with a probability p. Calculate the probability of failure of the whole memory.

7. Let $X_1,\ X_2 \ldots X_n$ be iid uniform RV over $[0, 1]$. Find the CDF of $M = max(X_1, X_2 \ldots X_n)$

8. Let c be a constant. Show that

$$Var(cX) = c^2 Var(X)$$
$$Var(c + X) = var(X)$$

9. If X is a Gaussian RV with mean μ and standard deviation σ, show that the moment generating function(MGF) is given by

$$E[e^{sX}] = e^{s\mu + \frac{s^2\sigma^2}{2}}$$

10. Use the MGF of a Gaussian RV from above and show that a linear sum of n independent Gaussian RVs is also Gaussian.

11. The normalized delay of an inverter is given by

$$D = D_{NOM} + 0.3\Delta L + 0.4\Delta V_{TH} \tag{1}$$

Where ΔL and ΔV_{TH} are normalized delay variations. Assume that in a particular technology, $D_{NOM} = 30$, ΔL and ΔV_{TH} are standard normal random variables. Calculate the fraction of inverters which have delays greater than 31.5. (This is called performance yield)

12. The leakage of an inverter is given by

$$I_{SUB} = I_0 e^{(-0.3\Delta L - 0.4\Delta V_{TH})}$$

Where ΔL and ΔV_{TH} are normalized delay variations. Assume that in a particular technology, $I_{NOM} = 100pA$, ΔL and ΔV_{TH} are standard normal random variables. Calculate the fraction of chips that leak more than $271.8pA$. (This is called leakage yield)

13. Use the MGF to prove central limit theorem(CLT) for iid RVs.

14. Let $Y = \sum_{i=1}^{n} \alpha_i X_i$ where X_i's are independent RVs with mean μ_i and standard deviation σ_i. Derive an expression for the mean and standard deviation of Y.

15. Sample mean of a random variable X is defined to be $S_N = \frac{1}{N}\sum_{i=1}^{N} X_i$ where X_i's are iid realizations of X. Compute the mean and variance of S_N. What is the distribution of S_N? For what value of N will S_N be equal to the actual mean μ?

16. Let Z be a standard normal variable (zero mean and unit variance). If $\{P(Z > z_\alpha\}) = \alpha$, show that $P(\{\frac{\sqrt{N}}{\sigma}(S_N - \mu)\} \leq z_{\frac{\alpha}{2}}) = 1 - \alpha$. The interval $\mu \pm \frac{z_{\frac{\alpha}{2}}\sigma}{\sqrt{N}}$ is called the $100(1 - \alpha)$ percent confidence estimator of the sample mean. Here S_N is as defined in the previous problem

Simulation

1. The delay in Equation (1) can be written as

$$D = 30 + 0.5X$$

where X is a standard normal random variable. In MATLAB/ OCTAVE use the "randn"function to generate N samples of X. $P(D > \alpha)$ can be evaluated as the fraction of values of D that exceed α. Now evaluate the following by varying N from 10^3 to 10^7 in geometrically progressing steps of $2X$

- $P(D > 30.5)$
- $P(D > 31.5)$
- $P(D > 32)$

Repeat the experiment five times and plot the above probabilities vs N. What do you observe?

2. Use the theory of importance sampling and show how you can reduce the required number of samples. What's the new PDF that you constructed?