Gyrotropic Materials

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When a material response is due to a magnetic field, it is called a gyrotropic material. An example of such a material is an electron gas (with neutralizing, stationary positive charges).

\[ \nabla \left( \nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} = -j \omega \mu_0 \left( \vec{j} + j \omega \cdot \vec{E} \right) \]  
(1)

We assume that all the dielectric properties come as a result of electron response, and set \( \varepsilon = \varepsilon_0 \). Guessing plane wave solutions, we assume

\[ \vec{E} = E_0 e^{j(\omega t - \vec{k} \cdot \vec{r})} \]

to get

\[ -j \vec{k} \left( -j \vec{k} \cdot \vec{E} \right) + k^2 \vec{E} = -j \omega \mu_0 \vec{j} + \omega^2 \mu_0 \varepsilon_0 \vec{E} \]

To make progress, we need to connect \( \vec{j} \) and \( \vec{E} \). Each electron obeys a force law:

\[ \frac{d\vec{v}}{dt} = \frac{q}{m} \left( \vec{E} + \vec{v} \times \vec{B} \right) \approx \frac{q}{m} \left( \vec{E} + \vec{v} \times \vec{B}_0 \right) \]

where \( \vec{B}_0 \) is an applied, uniform, magnetic field. This field is assumed to be so strong that the wave magnetic field’s contributions can be neglected. The current density is given by \( \vec{j} = n q \vec{v} \). To obtain an equation for \( \vec{j} \), we write the fluid equation for the flux of momentum, \( m \vec{u} \), where \( \vec{u} \) is the drift velocity. This equation is

\[ \frac{\partial \vec{n} \vec{u}}{\partial t} + \vec{u} \cdot \nabla (n \vec{u}) = \frac{qn}{m} \left( \vec{E} + \vec{u} \times \vec{B} \right) \approx \frac{qn}{m} \left( \vec{E} + \vec{u} \times \vec{B}_0 \right) = \frac{qn}{m} \vec{E} + \frac{1}{m} \vec{j} \times \vec{B} \]

Note that we have neglected pressure in this equation since that leads to a much more complex equation. We have also neglected friction. This equation now becomes an equation for \( \vec{j} \)

\[ \frac{\partial \vec{j}}{\partial t} + \vec{u} \cdot \nabla \vec{j} = \omega_p^2 \varepsilon_0 \vec{E} + \vec{j} \times \Omega \]

where \( \omega_p^2 = nq^2/m \omega_0 \) is called the plasma frequency (squared) and \( \Omega = qB_0/m \) is known as the cyclotron frequency. We assume that the drift velocity is small enough that the second (nonlinear) term can be neglected. That gives us the basic equation that connects current density and field:

\[ j \omega \vec{j} - \vec{j} \times \Omega = \omega_p^2 \varepsilon_0 \vec{E} \]

We immediately specialize to a very simple version of this problem, namely that of transverse waves. Then \( \vec{k} \) is along \( \hat{z} \), and \( \vec{E} \) and \( \vec{j} \) are in the \( x-y \) plane. Crossing the equation again with \( \Omega \) yields

\[ j \omega \vec{j} \times \Omega + \vec{j} \Omega^2 = \omega_p^2 \varepsilon_0 \vec{E} \times \Omega \]

Substituting for the cross product term involving \( \vec{j} \) from the previous equation yields

\[ j \omega \left( j \omega \vec{j} - \omega_p^2 \varepsilon_0 \vec{E} \right) + \vec{j} \Omega^2 = \omega_p^2 \varepsilon_0 \vec{E} \times \Omega \]

i.e.,

\[ \left( \Omega^2 - \omega^2 \right) \vec{j} = \omega_p^2 \varepsilon_0 \left( j \omega \vec{E} + \vec{E} \times \Omega \right) \]

Substituting into the wave equation yields

\[ -j \vec{k} \left( -j \vec{k} \cdot \vec{E} \right) + k^2 \vec{E} = -j \omega \mu_0 \varepsilon_0 \omega_p^2 \frac{\omega^2}{\Omega^2 - \omega^2} \left( j \omega \vec{E} + \vec{E} \times \Omega \right) + \omega^2 \mu_0 \varepsilon_0 \vec{E} \]

i.e.,

\[ -j \vec{k} \left( -j \vec{k} \cdot \vec{E} \right) + k^2 \vec{E} = \frac{\omega_p^2}{c^2 \Omega^2 - \omega^2} \left( \omega^2 \vec{E} - j \omega \vec{E} \times \Omega \right) + \omega^2 \vec{E} \]

The structure is now slightly different from the anisotropic case, in that we get
\[-\overrightarrow{k}k + k^2I - \omega^2jk_0\overrightarrow{E}\] \cdot \overrightarrow{E} = 0

where
\[\varepsilon = \varepsilon_0 \begin{pmatrix} A & jB & 0 \\ -jB & A & 0 \\ 0 & 0 & C \end{pmatrix}\]

(We did not derive the \(\varepsilon_{33} = \varepsilon_0C\) term, but it is there).

Written out the wave equation looks like this:
\[
\begin{pmatrix}
-k_0^2 + k^2 - k_0^2A & -k_0k_x - jk_0^2B & -k_0k_z \\
-k_0k_x + jk_0^2B & k^2 - k_0^2A & -k_0k_z \\
-k_0k_z & -k_0k_y & -k_0^2 + k^2 - k_0^2C
\end{pmatrix}
\begin{pmatrix}\overrightarrow{E}_x \\ \overrightarrow{E}_y \\ \overrightarrow{E}_z\end{pmatrix} = 0
\] (2)

The eigen solutions to this problem are got by finding the zeros of the determinant. We again rotate the coordinates so that \(\overrightarrow{k} = k_x\hat{x} + k_z\hat{z}\). In the absence of \(k_y\) the matrix becomes even simpler, and we finally get
\[
A \cdot \overrightarrow{E} = \begin{pmatrix}
-k_0^2 + k^2 - k_0^2A & -jk_0^2B & -k_0k_z \\
k_0^2B & k^2 - k_0^2A & 0 \\
-k_0k_z & 0 & -k_0^2 + k^2 - k_0^2C
\end{pmatrix}
\begin{pmatrix}\overrightarrow{E}_x \\ \overrightarrow{E}_y \\ \overrightarrow{E}_z\end{pmatrix} = 0
\] (3)

This problem is too complex to be treated here. We specialize to the case of axial \(\overrightarrow{k} = k\hat{z}\). Then, the equation becomes
\[
A \cdot \overrightarrow{E} = \begin{pmatrix}
k^2 - k_0^2A & -jk_0^2B & 0 \\
jk_0^2B & k^2 - k_0^2A & 0 \\
0 & 0 & -k_0^2 + k^2 - k_0^2C
\end{pmatrix}
\begin{pmatrix}\overrightarrow{E}_x \\ \overrightarrow{E}_y \\ \overrightarrow{E}_z\end{pmatrix} = 0
\] (4)

The determinant becomes
\[-k_0^2C [(k^2 - k_0^2A)^2 - k_0^2B^2] = 0\]

Clearly the roots are
\[k_0^2 = 0\]

(which is the DC electrostatic mode) and
\[k^2 = k_0^2(A \pm B)\]

These modes correspond to right and left circularly polarized modes, since at these values of \(k^2\), the matrix equation reduces to
\[
A \cdot \overrightarrow{E} = \begin{pmatrix}\alpha & \pm j\alpha \\ j\alpha & \alpha \\ \alpha & \alpha \end{pmatrix}
\begin{pmatrix}\overrightarrow{E}_x \\ \overrightarrow{E}_y \\ \overrightarrow{E}_z\end{pmatrix} = 0
\]

Clearly the solution has \(E_x\) and \(E_y\) that are equal in magnitude but \(\pi/2\) out of phase, i.e., a circularly polarized mode. The modes have different phase velocities
\[\frac{ck_0}{k} = \frac{1}{\sqrt{A \pm B}}\]

**Faraday Rotation**

Suppose a linearly polarized wave with its Electric field oriented along \(\hat{x}\) arrives at such a medium along the \(\hat{z}\) axis. The transmitted wave is given by
\[
\begin{pmatrix} E_0 \\ 0 \end{pmatrix} = \frac{E_0}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} & j/\sqrt{2} \end{pmatrix} e^{i(\omega - k^+z)} + \frac{E_0}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} & j/\sqrt{2} \end{pmatrix} e^{i(\omega - k^-z)}
\]

At the exit from the material, the transmitted wave now looks like (with \(z = L\))
\[
\left[\frac{E_0 e^{jk^+L}}{\sqrt{2}} \left( \frac{1/\sqrt{2}}{j/\sqrt{2}} \right) + \frac{E_0 e^{jk^-L}}{\sqrt{2}} \left( \frac{1/\sqrt{2}}{-j/\sqrt{2}} \right)\right] e^{i\omega L}
\]

We pull out a factor of \(e^{ik^+L}\), where \(k = 0.5(k^+ + k^-)\). Then,
\[
\left[\frac{E_0 e^{-j\Delta}}{\sqrt{2}} \left( \frac{1/\sqrt{2}}{j/\sqrt{2}} \right) + \frac{E_0 e^{j\Delta}}{\sqrt{2}} \left( \frac{1/\sqrt{2}}{-j/\sqrt{2}} \right)\right] e^{i(\omega - \pi L)}
\]

where \(\Delta = (k^+ - k^-)L\). This expression can be simplified:
\[|E_0 \hat{x} \cos \Delta - E_0 \hat{y} \sin \Delta| e^{i(\omega - \pi L)}\]

What this expression says is that the wave that emerges into the air has \(E_x\) and \(E_y\) in phase, but with an Electric Field vector that now points along a different direction in the \(x - y\) plane. This rotation of a linearly polarized wave, leaving it linearly polarized, is called Faraday rotation.