Reflection and Transmission at a Dielectric Boundary - Oblique Case

12th March 2007

We consider a wave incident on a dielectric slab oriented along the $x$-$z$ plane. The wave vector makes an angle of $\theta_i$ with the normal to the surface. We wish to compute the resulting waves.

The wave travels in the $x$-$y$ plane with its Electric Field given by

$$\vec{E}_0 e^{i(\omega t - k_x x - k_y y)}$$

where $k_x = k \sin \theta_i$ and $k_y = k \cos \theta_i$. The direction of the Electric Field is in the plane normal to $\vec{k}$, i.e., $\vec{E}_0 = E_0 \hat{z} + E_{0\perp} (\cos \theta_i \hat{x} + \sin \theta_i \hat{y})$. The magnetic field is given by

$$\vec{H} = \frac{\vec{k} \times \vec{E}_0}{\omega \mu} = \frac{1}{\eta} (E_{0\perp} \hat{z} + E_{0\parallel} (-\cos \theta_i \hat{x} - \sin \theta_i \hat{y}))$$

We solve the problem in two parts:
1. Consider $E_0 \perp \zhat = 0$. Then the Electric Field is purely along $\zhat$ and is purely tangential to the dielectric surface. The magnetic field is partly normal, and partly tangential. The tangential component is $-E_0 \cos \theta / \eta$.

2. Consider $E_0 = 0$. Then the Magnetic Field is purely along $\zhat$ and is purely tangential to the dielectric surface. The Electric field is now partly normal and partly tangential. The tangential component of $\vec{E}$ is $E_0 \perp \cos \theta i$. 

If we can solve these two problems, than the general solution is merely a linear combination of these two problems, since Maxwell’s Equations are linear integro-differential equations. (Of course the $\vec{v} \times \vec{B}$ term and the $\vec{v} \cdot \vec{v}$ term in the momentum equation of fluids are nonlinear, but we ignore such distractions here.) These two primary solutions are called Polarizations. The first can be called Transverse Electric Mode since the Electric Field is purely tangential to the surface and the second can be called Transverse Magnetic Mode. They are given different names, but the meaning is always clear from the context.

### The Tangential Electric Field Case

\[
\begin{align*}
\vec{E}_t &= E_0 \hat{z} e^{il(\omega - k_x x - k_y y)} \\
\vec{H}_t &= \frac{E_0}{\eta_1} (-\cos \theta i \hat{x} - \sin \theta i \hat{y}) e^{il(\omega - k_x x - k_y y)}
\end{align*}
\]

As we argued in class, the reflected and transmitted waves must have the same frequency and the reflected wave must make the same angle as the incident wave, due to the need to match across the boundary. Thus

\[
\begin{align*}
\vec{E}_r &= E_r \hat{z} e^{il(\omega - k_x x - k_y y)} \\
\vec{H}_r &= \frac{E_r}{\eta_1} (-\cos \theta r i \hat{x} - \sin \theta r i \hat{y}) e^{il(\omega - k_x x - k_y y)}
\end{align*}
\]

where the sign of $k_y y$ has been flipped to show that the wave is travelling back, and the magnetic field is now perpendicular to the new $\vec{k}$.

The transmitted wave has the form

\[
\begin{align*}
\vec{E}_t &= E_t \hat{z} e^{il(\omega - k_x x - k'_y y)} \\
\vec{H}_t &= \frac{E_t}{\eta_2} (-\cos \theta r i \hat{x} - \sin \theta r i \hat{y}) e^{il(\omega - k_x x - k'_y y)}
\end{align*}
\]

Note that both $\omega$ and $k_x$ are the same for all three waves, since they must agree along the interface.

We now impose continuity of the tangential Electric and Magnetic Fields at the interface. That yields

\[
\begin{align*}
E_0 + E_r &= E_t \\
\frac{\cos \theta_i}{\eta_1} (-E_0 + E_r) &= -\frac{\cos \theta_r}{\eta_2} E_t \\
\text{i.e.,} \quad E_0 - E_r &= \frac{\eta_1 \cos \theta_r}{\eta_2 \cos \theta_i} E_t
\end{align*}
\]
Now,
\[
\frac{\sin \theta_i}{\sin \theta_r} = \frac{k_x / k_1}{k_x / k_2} = \frac{k_2}{k_1} = \frac{c_1}{\varepsilon_2} = \sqrt{\varepsilon_2 / \varepsilon_1} = n_2
\]
which is Snell’s Law. Let
\[
\eta_1 \cos \theta_i = n_2 \cos \theta_r = \frac{\tan \theta_i}{\tan \theta_r} = \alpha
\]
In terms of this quantity, the solution is:
\[
E_t = \frac{2}{1 + \alpha} E_0
\]
\[
E_r = \frac{1 - \alpha}{1 + \alpha} E_0
\]
There is always reflection and there is always transmission of this polarization.
\[
R = \left| \frac{E_r}{E_0} \right|^2 = \left( \frac{1 - \alpha}{1 + \alpha} \right)^2
\]
\[
T = \left| \frac{E_t}{E_0} \right|^2 \frac{\eta_2 \cos \theta_i}{\eta_1 \cos \theta_r} = \frac{4\alpha}{(1 + \alpha)^2}
\]
Clearly, \( R + T = 1 \) as expected.

**The Tangential Magnetic Field Case**

The incident wave is
\[
\vec{E}_i = E_0 (\cos \theta_i \hat{x} + \sin \theta_i \hat{y})
\]
\[
\vec{H}_i = \frac{E_0}{\eta_1} \hat{z}
\]
The reflected wave is
\[
\vec{E}_r = E_r (-\cos \theta_i \hat{x} + \sin \theta_i \hat{y})
\]
\[
\vec{H}_r = \frac{E_r}{\eta_1} \hat{z}
\]
The transmitted wave is
\[
\vec{E}_t = E_t (\cos \theta_r \hat{x} + \sin \theta_r \hat{y})
\]
\[
\vec{H}_t = \frac{E_t}{\eta_2} \hat{z}
\]
Continuity of the tangential components now gives
\[
E_0 - E_r = \frac{\cos \theta_i}{\cos \theta_r} E_t
\]
\[
(E_0 + E_r) = \frac{\eta_1}{\eta_2} E_t
\]
Subtracting the two equations we get \( E_r \):

\[
E_r = \frac{1}{2} \left( \frac{\eta_1}{\eta_2} \cos \theta_r \right) E_r
\]

\[
E_0 = \frac{1}{2} \left( \frac{\eta_1}{\eta_2} + \frac{\cos \theta_r}{\cos \theta_i} \right) E_i
\]

from which the all the fields can be calculated.

Suppose \( \varepsilon_2 > \varepsilon_1 \), i.e., \( \eta_1 > \eta_2 \). For normal incidence,

\[
\frac{\eta_1}{\eta_2} \cos \theta_r = \frac{\eta_1}{\eta_2} - 1 > 0
\]

For grazing incidence,

\[
\frac{\eta_1}{\eta_2} \cos \theta_r = \frac{\eta_1}{\eta_2} - \infty < 0
\]

Hence there is an angle at which \( E_r \) becomes zero. This is called the Brewster angle. TM light incident at this angle suffers no reflections. Thus the reflected light at this angle is purely TE.