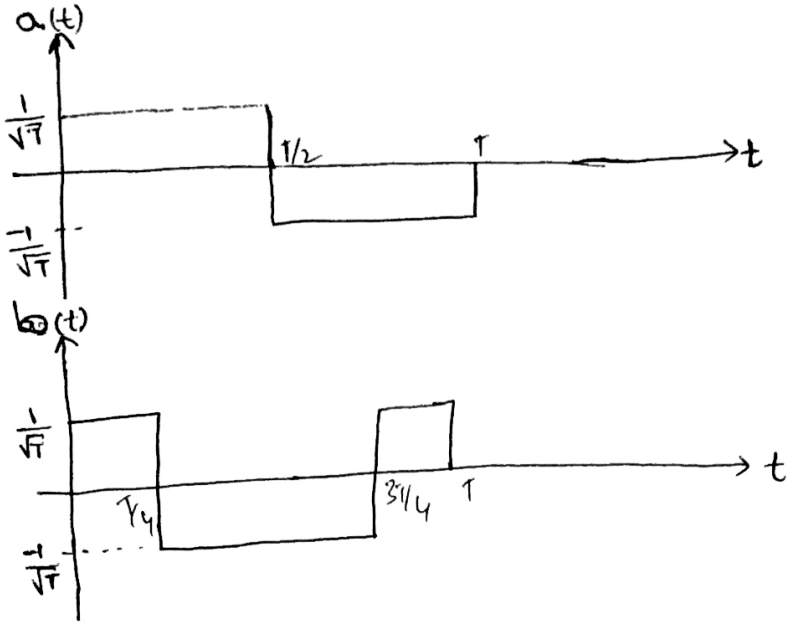
TUTORIAL - 1

SOLUTIONS

1A) By observation, we can say the orthonormal bases will be,



$s_1(t) = \sqrt{T}\, b_1(t)$

$s_2(t) = \sqrt{T}\, a(t)$

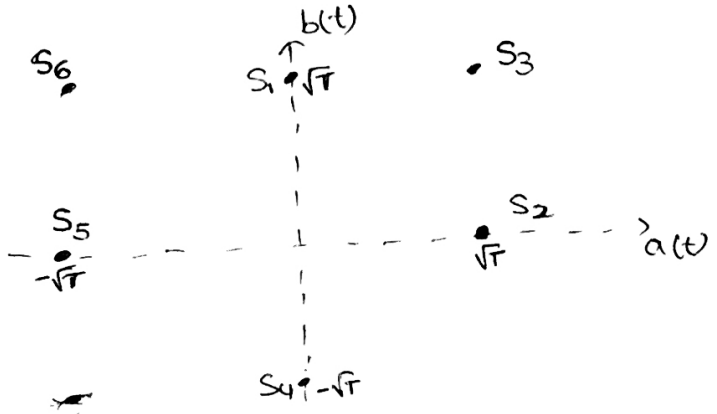
$s_3(t) = s_1 + s_2 = \sqrt{T}(a(t)) + \sqrt{T}\, b(t)$

$s_4(t) = -s_1(t)$

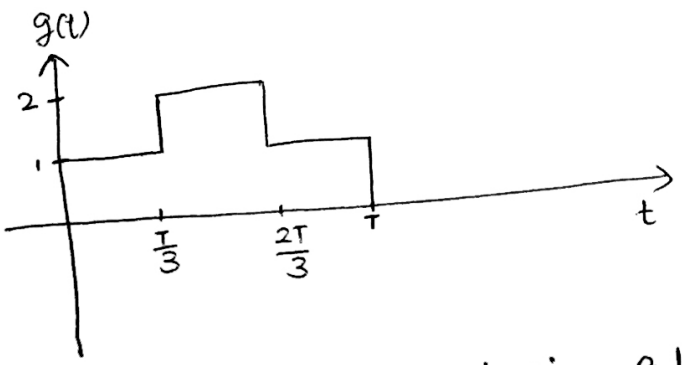
$s_5(t) = -s_2(t)$

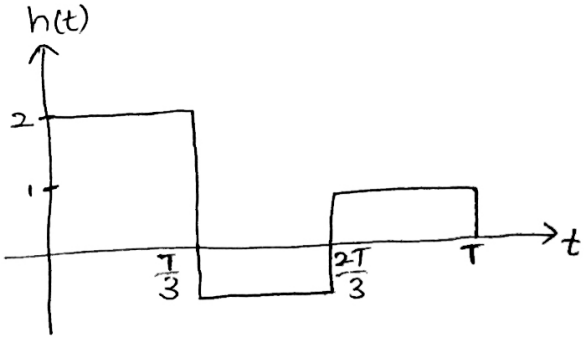
$s_6(t) = s_1 - s_2$

$= \sqrt{T} b(t) - \sqrt{T}\, a(t)$

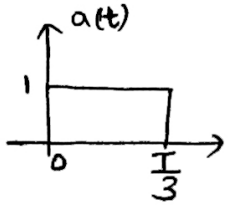


2A)



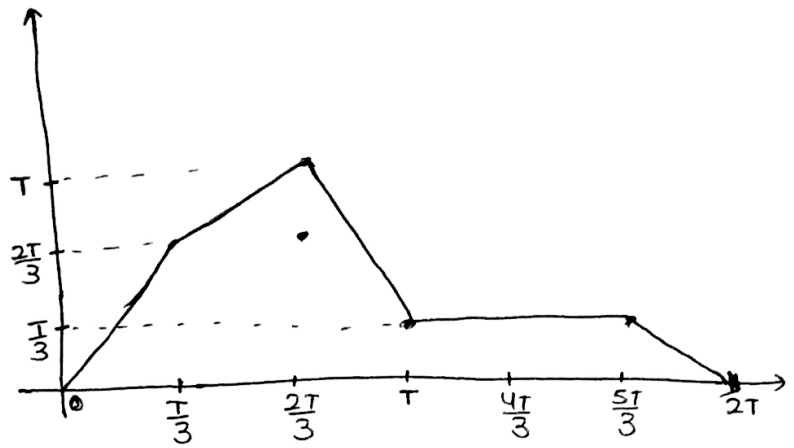


Let's say the basic pulse is

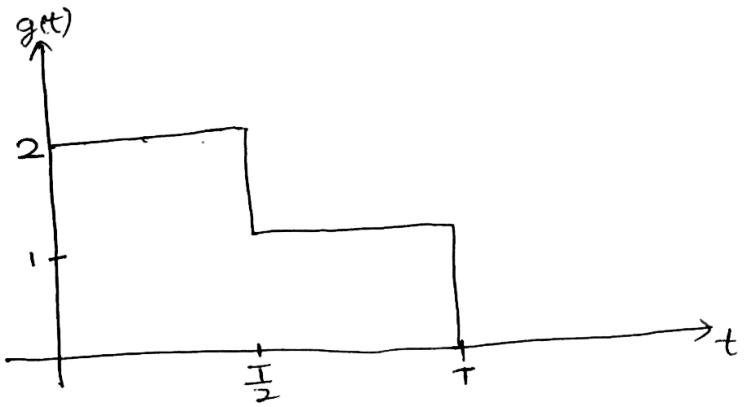


$g(t) = a(t) + 2a\left(t-\frac{T}{3}\right) + a\left(t-\frac{2T}{3}\right)$

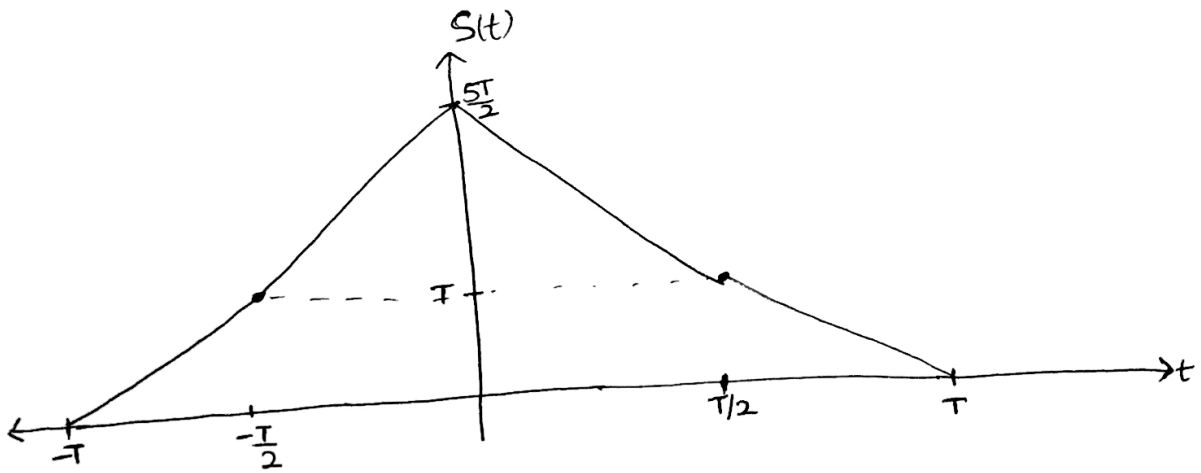
Matched filter is $g(t) * h(t)$



3A)

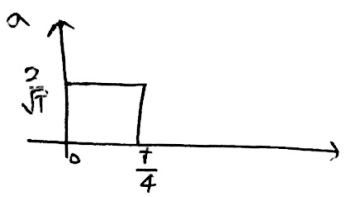


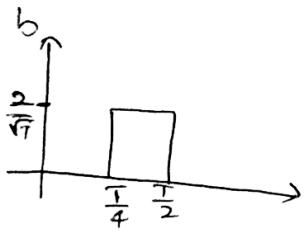
$$S(t) = \int g(\tau)\, g(\tau - t)\, d\tau$$

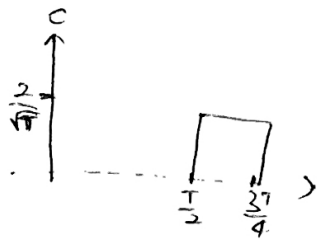


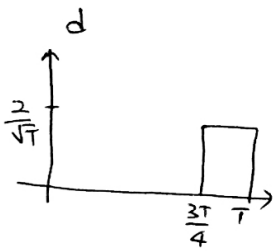
4A)

4A) Basis functions are







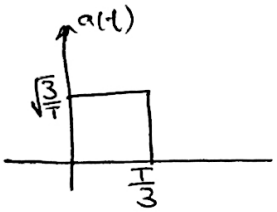


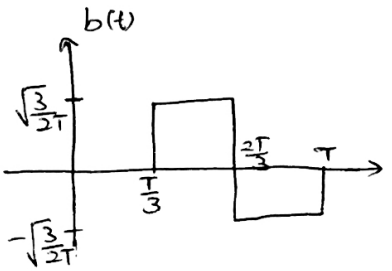
$$\begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{\text{Full rank matrix}} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

So, this is the compact basis.

5A) Basis are

(a)





(b)

$S_1 = \sqrt{\frac{T}{3}}\, a - \sqrt{\frac{2T}{3}}\, b$

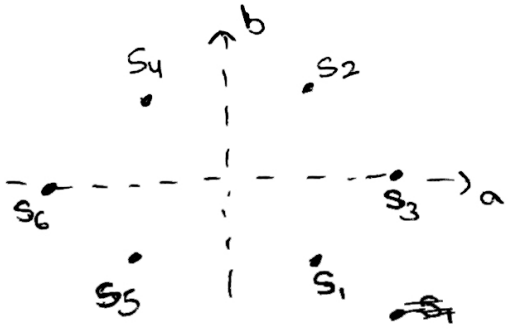
$S_2 = \sqrt{\frac{T}{3}}\, a + \sqrt{\frac{2T}{3}}\, b$

$S_3 = 2\sqrt{\frac{T}{3}}\, a$

$S_4 = -\sqrt{\frac{T}{3}}\, a + \sqrt{\frac{2T}{3}}\, b$

$S_5 = -\sqrt{\frac{T}{3}}\, a - \sqrt{\frac{2T}{3}}\, b$

$S_6 = -2\sqrt{\frac{T}{3}}\, a$



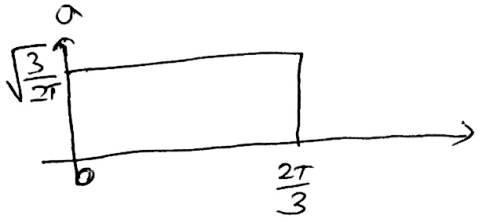
(4)

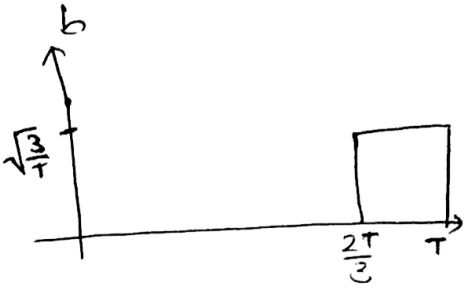
(c) $E_a = \frac{1}{6}\left[2\left(\frac{4T}{3}\right) + 4(T)\right]$

$= \left(\frac{1}{6}\right)\left[\frac{20T}{3}\right] = \frac{10T}{9}$

Minimum distance $= \sqrt{T} = \sqrt{\frac{9E_a}{10}}$

6A) Basis functions are



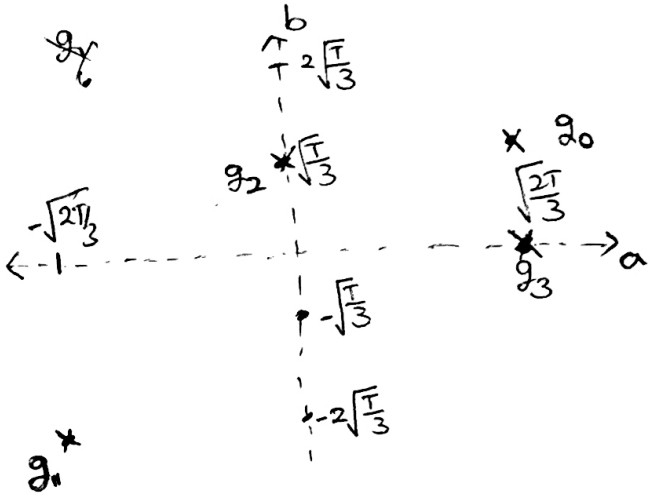


$g_0 = \sqrt{\frac{2T}{3}}\,a + \sqrt{\frac{T}{3}}\,b$

$g_1 = -\sqrt{\frac{2T}{3}}\,a - 2\sqrt{\frac{T}{3}}\,b$

$g_2 = \sqrt{\frac{T}{3}}\,b$

$g_3 = \sqrt{\frac{2T}{3}}\,a$



Nearest distance $= \sqrt{\frac{T}{3}}$

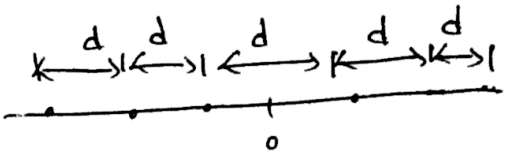
(between g_0 and g_3) $\Rightarrow$ Multiplicity $= 1$

7A)

7.1

given

$$s_m(t) = s_m \psi(t), \quad m = 1, 2, \ldots, M \qquad 0 \le t \le T$$



where $s_m = \sqrt{\mathcal{E}_g}\, A_m, \quad m = 1, 2, \ldots M$

For equally probable signals, average energy is

$$\mathcal{E}_{av} = \frac{1}{M} \sum_{m=1}^{M} \mathcal{E}_m = \frac{\mathcal{E}_g}{M} \sum_{m=1}^{M} A_m^2$$

If the signal amplitudes are symmetric about the origin, then

$$A_m = (2m - 1 - M), \quad m = 1, 2, \ldots M$$

$\therefore$ Avg energy $\mathcal{E}_{av} = \frac{\mathcal{E}_g}{M} \sum_{m=1}^{M} (2m-1-M)^2$

$$= \frac{\mathcal{E}_g}{M} \sum_{m=1}^{M} [4m^2 + (M+1)^2 - 4m(M+1)]$$

$$= \frac{\mathcal{E}_g}{M} \left(4 \sum_{m=1}^{M} m^2 + M(M+1)^2 - 4(M+1) \sum_{m=1}^{M} m \right)$$

$$= \frac{\mathcal{E}_g}{M} \left(4 \frac{M(M+1)(2M+1)}{6} + M(M+1)^2 - 4(M+1)\frac{M(M+1)}{2} \right)$$

$$= \frac{\mathcal{E}_g}{M} \left(\frac{M(M^2-1)}{3} \right) = \mathcal{E}_g (M^2-1)/3 \; //.$$

7.2

Correlation coefficient of two ~~adjacent~~ signal points m and n can be given by

$$\gamma_{mn} = \frac{s_m \cdot s_n}{|s_m| |s_n|} \quad \text{where } s_m = (s_{m1}, s_{m2}, \ldots s_{mN}) \text{ and } s_{mj} = \pm \sqrt{\frac{\mathcal{E}_s}{N}}$$

In case of two adjacent signal points, it differ in only one coordinate and hence

$$s_m \cdot s_n = \sum_{j=1}^{N} s_{mj}\, s_{nj} = \sum_{j \neq k} s_{mj} s_{nj} + s_{mk} s_{nk}$$

$$= (N-1)\frac{\varepsilon_s}{N} - \frac{\varepsilon_s}{N} = \frac{N-2}{N}\varepsilon_s$$

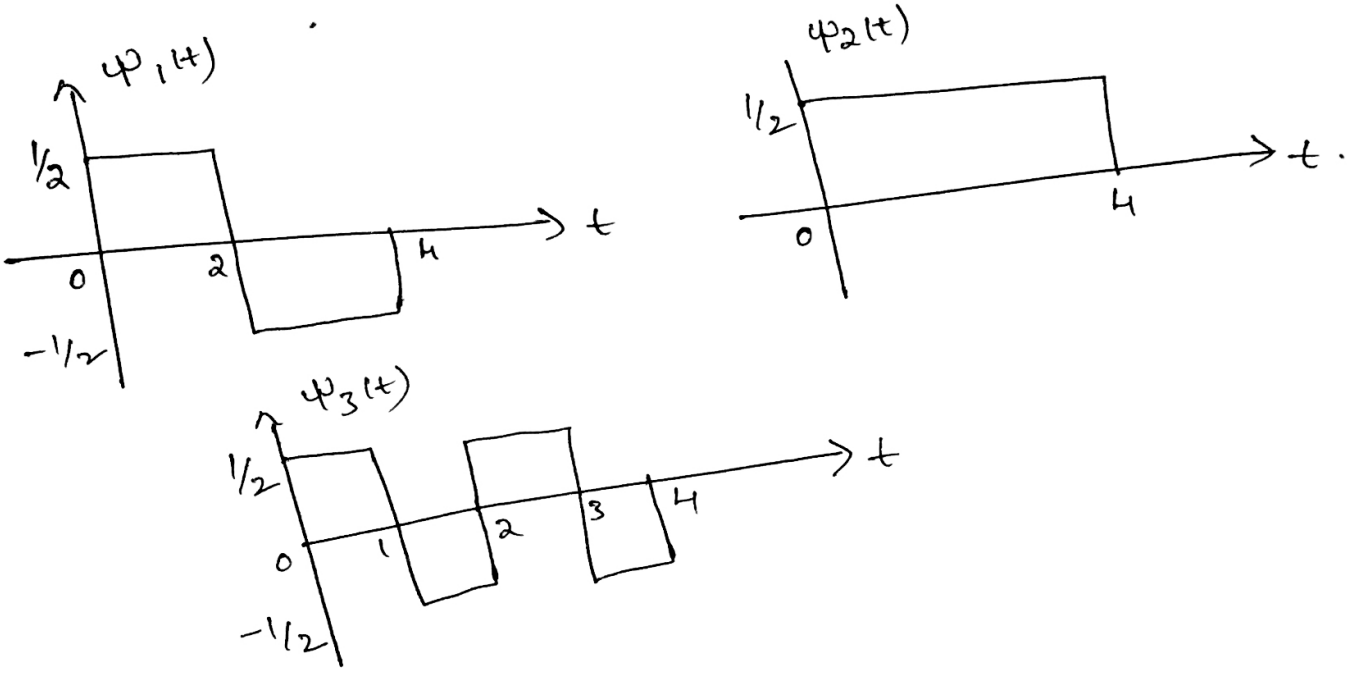
Also $|s_m| = |s_n| = \sqrt{\varepsilon_s}$

$$\therefore \rho_{mn} = \frac{N-2}{N} \frac{\varepsilon_s}{\sqrt{\varepsilon_s}\cdot\sqrt{\varepsilon_s}} = \frac{N-2}{N} \;//$$

The euclidean distance $d = \sqrt{|s_m - s_n|^2}$

$$= \sqrt{\left|\mp 2\sqrt{\frac{\varepsilon_s}{N}}\right|^2} = \sqrt{4\frac{\varepsilon_s}{N}} = 2\sqrt{\frac{\varepsilon_s}{N}}$$

7.3



1) The waveforms are orthogonal if and only if

$$\int_{-\infty}^{\infty} \psi_m(t)\, \psi_n(t)\, dt = 0, \quad m \neq n$$

$$\therefore \rightarrow \int_{-\infty}^{\infty} \psi_1(t)\, \psi_2(t)\, dt = \int_0^4 \psi_1(t)\, \psi_2(t)\, dt$$

$$= \int_0^2 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) dt + \int_2^4 \left(\frac{-1}{2}\right)\left(\frac{1}{2}\right) dt = 0$$

$$\rightarrow \int_{-\infty}^{\infty} \psi_1(t)\,\psi_3(t)\,dt = \int_0^4 \psi_1(t)\,\psi_3(t)\,dt$$

$$= \int_0^1 (1/2)(1/2)\,dt + \int_1^2 (1/2)(-1/2)\,dt + \int_2^3 (-1/2)(1/2)\,dt + \int_3^4 (-1/2)(-1/2)\,dt = 0$$

$$\rightarrow \int_{-\infty}^{\infty} \psi_2(t)\,\psi_3(t)\,dt = \int_0^4 \psi_2(t)\,\psi_3(t)\,dt$$

$$= \int_0^1 (1/2)(1/2)\,dt + \int_1^2 (1/2)(-1/2)\,dt + \int_2^3 (1/2)(1/2)\,dt + \int_3^4 (1/2)(-1/2)\,dt = 0$$

Hence the waveforms are orthogonal.

2) weighting coefficients $x_n = \int_{-\infty}^{\infty} x(t)\,\psi_n(t)\,dt$, $n=1,2,3$.

$$x_1 = \int_0^4 x(t)\,\psi_1(t)\,dt = \int_0^1 (-1)(1/2)\,dt + \int_1^2 (1)(1/2)\,dt + \int_2^3 (1)(-1/2)\,dt + \int_3^4 (-1)(-1/2)\,dt = 0$$

Similarly

$x_2 = 0$ and $x_3 = 0$.

$\therefore$ $x(t)$ is orthogonal to the signal $\psi_n(t)$ and thus it cannot be expressed as its linear combinations.

7.4

given

$$x(t) = \sin(\pi t/4) \quad ; \quad 0 \le t \le 4$$

Now $\hat{x}(t) = \sum_{n=1}^{3} c_n \psi_n(t)$

1) The expansion coefficients $\{c_n\}$ that minimize the mean square error should satisfy

$$c_n = \int_{-\infty}^{\infty} x(t)\,\psi_n(t)\,dt = \int_0^4 \sin\left(\frac{\pi t}{4}\right)\psi_n(t)\,dt$$

$$\therefore \quad c_1 = \int_0^4 \sin\left(\frac{\pi t}{4}\right)\psi_1(t)\,dt = \int_0^2 \sin\left(\frac{\pi t}{4}\right)\cdot\left(\tfrac{1}{2}\right)dt + \int_2^4 \sin\left(\frac{\pi t}{4}\right)\left(-\tfrac{1}{2}\right)dt$$

$$= -\frac{2}{\pi}\left(\cos\frac{\pi t}{4}\right)_0^2 + \frac{2}{\pi}\left(\cos\frac{\pi t}{4}\right)_2^4 = 0$$

$$c_2 = \int_0^4 \sin\left(\frac{\pi t}{4}\right)\psi_2(t)\,dt = \left(\frac{1}{2}\right)\int_0^4 \sin\left(\frac{\pi t}{4}\right)dt$$

$$= -\frac{2}{\pi}\left(\cos\frac{\pi t}{4}\right)_0^4 = \frac{4}{\pi}$$

$$c_3 = \int_0^4 \sin\left(\frac{\pi t}{4}\right)\psi_3(t)\,dt = 0$$

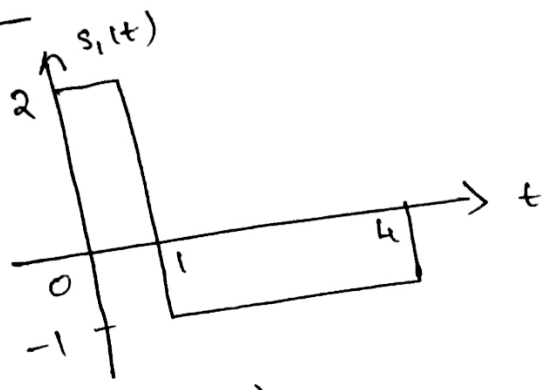
2) Residual mean square error

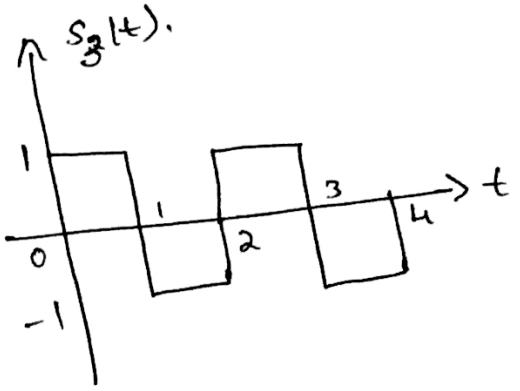
$$E_{min} = \int_{-\infty}^{\infty} |x(t)|^2\,dt - \sum_{i=1}^{3} |c_i|^2$$

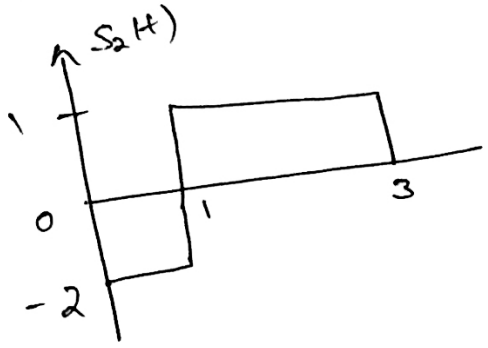
$$= \int_0^4 \left(\sin\frac{\pi t}{4}\right)^2 dt - \left(\frac{4}{\pi}\right)^2 = \int_0^4 \frac{1-\cos 2\left(\frac{\pi t}{4}\right)}{2}\,dt - \frac{16}{\pi^2}$$

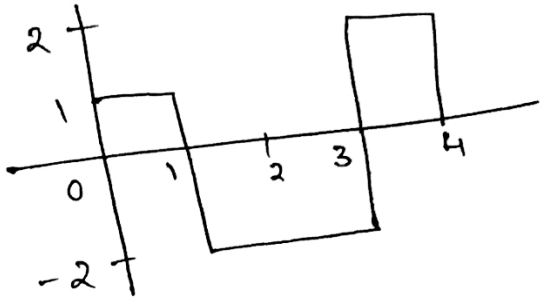
$$= 2 - \frac{1}{\pi}\left(\sin\frac{\pi t}{2}\right)_0^4 - \frac{16}{\pi^2} = 2 - \frac{16}{\pi^2} \;//$$

7.5









1) Let we consider the set of basis functions

$$\psi_1(t) = \begin{cases} 1 & ; 0 \le t < 1 \\ 0 & ; \text{otherwise} \end{cases} \qquad \psi_2(t) = \begin{cases} 1 & ; 1 \le t \le 2 \\ 0 & ; \text{otherwise} \end{cases}$$

$$\psi_3(t) = \begin{cases} 1 & ; 2 \le t < 3 \\ 0 & ; \text{otherwise} \end{cases} \qquad \psi_4(t) = \begin{cases} 1 & ; 3 \le t < 4 \\ 0 & ; \text{otherwise} \end{cases}$$

$$\therefore \begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \\ \psi_4(t) \end{pmatrix}.$$

Now by finding the rank of transformation matrix we could determine the dimensionality of the waveforms

$\therefore$

$$\begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to 2R_3 - R_1 \\ \longrightarrow \\ R_4 \to 2R_4 - R_1 \end{matrix}$$

$$\begin{pmatrix} 2 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 \\ 0 & -3 & -3 & 5 \end{pmatrix} \longrightarrow \text{interchange } R_2 \text{ with } R_3$$

$$\begin{pmatrix} 2 & -1 & -1 & -1 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & -3 & -3 & 5 \end{pmatrix} \longrightarrow R_4 \to R_4 - 3R_1$$

$$\begin{pmatrix} 2 & -1 & -1 & -1 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -12 & 8 \end{pmatrix} \longrightarrow \text{interchange } R_3 \text{ with } R_4$$

$$\begin{pmatrix} \boxed{2} & -1 & -1 & -1 \\ 0 & \boxed{-1} & 3 & -1 \\ 0 & 0 & \boxed{-12} & 8 \\ 0 & 0 & 0 & \boxed{-1} \end{pmatrix}$$

Therefore we have pivot ~~columns~~ elements in all 4 columns and hence

rank = dimensionality of waveform = 4 //

2) $s_1 = [2 \quad -1 \quad -1 \quad -1]$

$s_2 = [-2 \quad 1 \quad 1 \quad 0]$

$s_3 = [1 \quad -1 \quad 1 \quad -1]$

$s_4 = [1 \quad -2 \quad -2 \quad 2]$

3) $d_{12} = \sqrt{|s_1 - s_2|^2} = \sqrt{|[4 \quad -2 \quad -2 \quad -1]|^2} = \sqrt{25}$

$d_{13} = \sqrt{|s_1 - s_3|^2} = \sqrt{5}$

$d_{14} = \sqrt{|s_1 - s_4|^2} = \sqrt{12}$

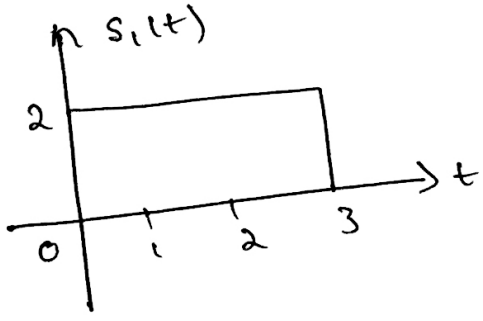
$d_{23} = \sqrt{|s_2 - s_3|^2} = \sqrt{14}$

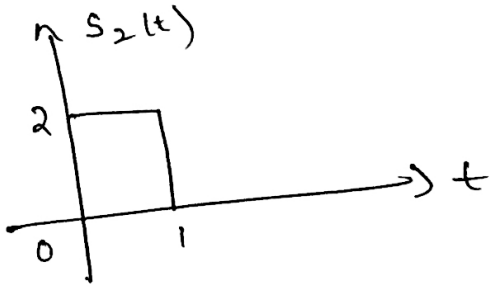
$d_{24} = \sqrt{|s_2 - s_4|^2} = \sqrt{31}$

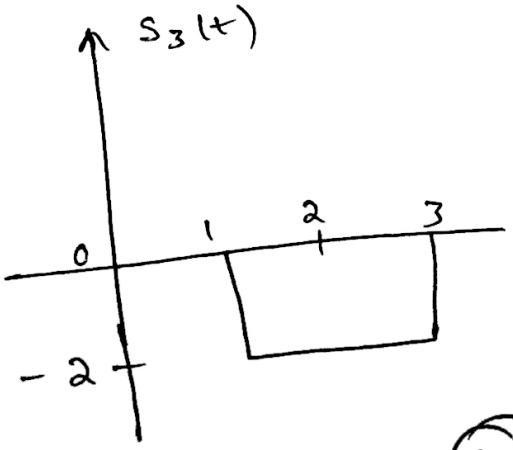
$d_{34} = \sqrt{|s_3 - s_4|^2} = \sqrt{19}$

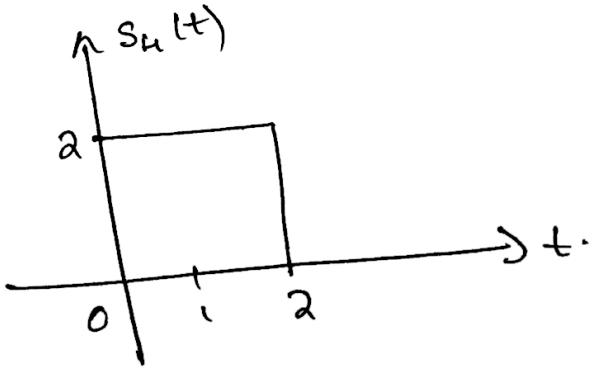
$\therefore$ minimum distance $= d_{min} = d_{13} = \sqrt{5}$ //

7.6









Let us consider a set of orthonormal functions

$$\psi_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \psi_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad s_1 = [2 \quad 2 \quad 2]$$

$$s_2 = [2 \quad 0 \quad 0]$$

$$s_3 = [0 \quad -2 \quad -2]$$

$$s_4 = [2 \quad 2 \quad 0]$$

7.7

The energy of $s_m'(t)$ is

$$\mathcal{E}' = \int_{-\infty}^{\infty} |s_m'(t)|^2 dt = \int_{-\infty}^{\infty} \left| s_m(t) - \frac{1}{m} \sum_{k=1}^{m} s_k(t) \right|^2 dt$$

$$= \int_{-\infty}^{\infty} s_m^2(t) + \frac{1}{m^2} \sum_{k=1}^{m} \sum_{l=1}^{m} \int_{-\infty}^{\infty} s_k(t)\, s_l(t)\, dt$$

$$- \frac{1}{m} \sum_{k=1}^{m} \int_{-\infty}^{\infty} s_m(t)\, s_k(t)\, dt - \frac{1}{m} \sum_{l=1}^{m} \int_{-\infty}^{\infty} s_m(t)\, s_l(t)\, dt$$

$$= \mathcal{E} + \frac{1}{m^2} \sum_{k=1}^{m} \sum_{l=1}^{m} \mathcal{E}\, \delta_{kl} - \frac{2}{m} \mathcal{E}$$

$$= \mathcal{E} + \frac{1}{m} \mathcal{E} - \frac{2}{m} \mathcal{E} = \left(\frac{m-1}{m} \right) \mathcal{E}$$

The correlation coefficient is given by

$$\rho_{mn} = \frac{\int_{-\infty}^{\infty} s_m'(t)\, s_n'(t)\, dt}{\left[\int_{-\infty}^{\infty} |s_m'(t)|^2 dt\right]^{1/2} \left[\int_{-\infty}^{\infty} |s_n'(t)|^2 dt\right]^{1/2}}$$

$$= \frac{1}{\mathcal{E}'} \int_{-\infty}^{\infty} \left(s_m(t) - \frac{1}{m}\sum_{k=1}^{m} s_k(t)\right)\left(s_n(t) - \frac{1}{m}\sum_{l=1}^{m} s_l(t)\, dt\right)$$

$$= \frac{1}{\mathcal{E}'}\left(\int_{-\infty}^{\infty} s_m(t)\, s_n(t)\, dt + \frac{1}{m^2}\sum_{k=1}^{m}\sum_{l=1}^{m}\int_{-\infty}^{\infty} s_k(t) s_l(t) dt\right)$$

$$- \frac{1}{\mathcal{E}'}\left(\frac{1}{m}\sum_{k=1}^{m}\int_{-\infty}^{\infty} s_n(t) s_k(t)\, dt + \frac{1}{m}\sum_{l=1}^{m}\int_{-\infty}^{\infty} s_m(t) s_l(t) dt\right)$$

$$= \frac{\frac{1}{m^2}\, m\mathcal{E} - \frac{1}{m}\mathcal{E} - \frac{1}{m}\mathcal{E}}{\left(\frac{m-1}{m}\right)\mathcal{E}} = \frac{-1}{m-1} \;//$$

7.8

Assuming that $E[n^2(t)] = \sigma_n^2$, we obtain

$$E[n_1 n_2] = E\left[\left(\int_0^T s_1(t)\, n(t)\, dt\right)\left(\int_0^T s_2(v)\, n(v)\, dv\right)\right]$$

$$= \int_0^T \int_0^T s_1(t)\, s_2(v)\, E[n(t)\, n(v)]\, dt\, dv$$

$$= \sigma_n^2 \int_0^T s_1(t)\, s_2(t)\, dt = 0 \;// \quad (\because s_1, s_2 \text{ are orthogonal})$$

7.9

1) The received s/g may be expressed as

$$r(t) = \begin{cases} n(t) & \text{if } s_0(t) \text{ was transmitted} \\ A + n(t) & \text{if } s_1(t) \text{ was transmitted} \end{cases}$$

Assuming $s(t)$ has unit energy, then the sampled o/p of the cross correlators are

$$r = s_m + n, \quad m = 0, 1.$$

where $s_0 = 0$, $s_1 = A\sqrt{T}$ and the noise term n is a zero-mean gaussian R.V with variance

$$\sigma_n^2 = E\left[\frac{1}{\sqrt{T}} \int_0^T n(t)\,dt \; \frac{1}{\sqrt{T}} \int_0^T n(\tau)\,d\tau \right]$$

$$= \frac{1}{T} \int_0^T \int_0^T E[n(t)\, n(\tau)]\,dt\,d\tau$$

$$= \frac{N_0}{2T} \int_0^T \int_0^T s(t-\tau)\,dt\,d\tau = \frac{N_0}{2}$$

The p.d.f of the sampled o/p is

$$f(r|s_0) = \frac{1}{\sqrt{\pi N_0}} e^{-r^2/N_0}$$

$$f(r|s_1) = \frac{1}{\sqrt{\pi N_0}} e^{-(r - A\sqrt{T})^2/N_0}$$

Since the s/g's are equally probable, the optimal detector decides in favor of s_0 if

$$PM(r, s_0) = f(r|s_0) > f(r|s_1) = PM(r, s_1)$$

otherwise it decides in favor of s_1. The decision rule may be expressed as

$$\frac{P_m(r, s_0)}{P_m(r, s_1)} = e^{\frac{(r-A\sqrt{T})^2 - r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \underset{s_1}{\overset{s_0}{\gtrless}} 1$$

or equivalently $\quad r \underset{s_0}{\overset{s_1}{\gtrless}} \frac{1}{2}A\sqrt{T}$

The optimum threshold is $\frac{1}{2}A\sqrt{T}$

2) Avg probability of error

$$P(e) = \frac{1}{2}P(e/s_0) + \frac{1}{2}P(e/s_1)$$

$$= \frac{1}{2}\int_{1/2 A\sqrt{T}}^{\infty} f(r/s_0)\,dr + \frac{1}{2}\int_{-\infty}^{1/2 A\sqrt{T}} f(r/s_1)\,dr$$

$$= \frac{1}{2}\int_{1/2 A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-r^2/N_0}\,dr + \frac{1}{2}\int_{-\infty}^{1/2 A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}}\,dr$$

$$= \frac{1}{2}\int_{1/2\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx + \frac{1}{2}\int_{-\infty}^{-1/2\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

$$= Q\left[\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}\right] = Q\left[\sqrt{SNR}\right]$$

where $SNR = \dfrac{\frac{1}{2}A^2T}{N_0}$

Thus, on-off signalling requires a factor of two more energy to achieve the same probability of error as the antipodal signalling.