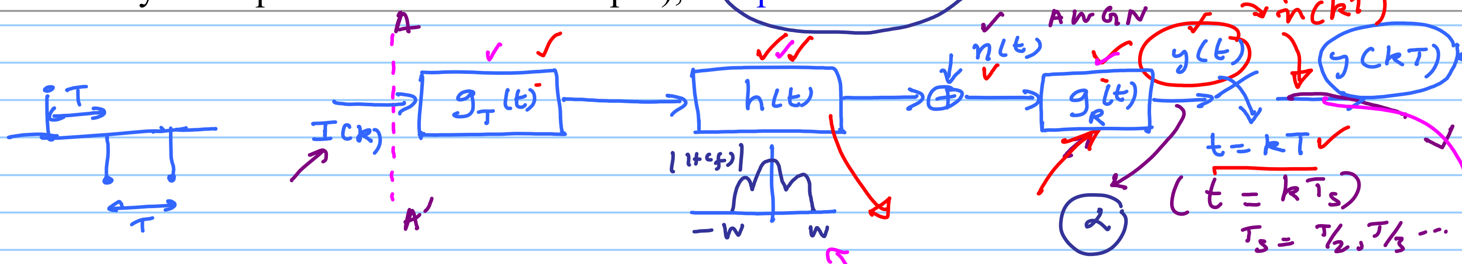
Lesson 4 -- Digital communications thro distorting channels – Sequence estimation, MLSE and the Viterbi algorithm for ISI channels (Sec. 8.6 and only some parts of Sec. 8.5 in Chap-8), and practical issues in MLSE



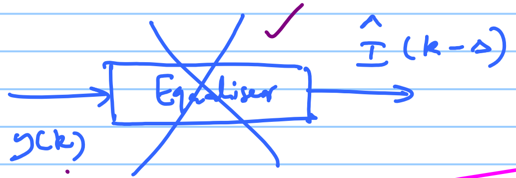
ISI present

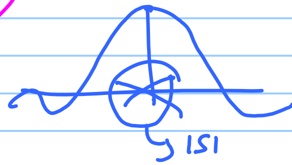
$E[\tilde{n}(k)\tilde{n}(m)] \neq 0$

$k \neq m$

$\underbrace{g_T(t) * h(t) * g_R(t)}_{\cancel{\ast}\ c(t)}$

Nyquist shape





$|f_0| > |f_1| > |f_2| \ldots$

$\{f_0, f_1, f_2 \ldots f_{L-1}\}$

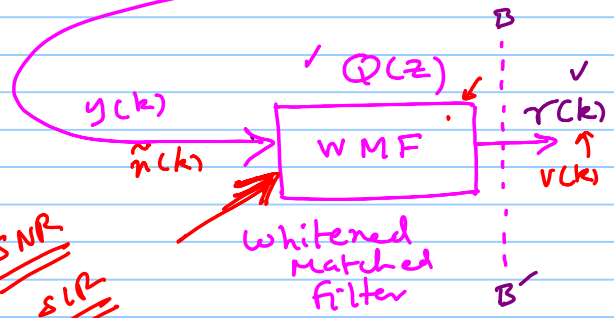
1972

Forney

Spectral Factorization

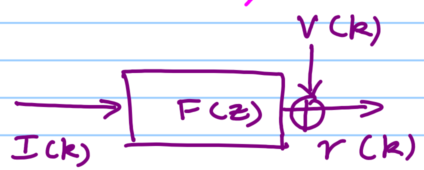
SNR

SIR



$F(z)$ → FIR (of length L)

→ Minimum phase filter



$$r(k) = \sum_{l=0}^{L-1} f_l I(k-l) + v(k)$$

Band-limited AWGN

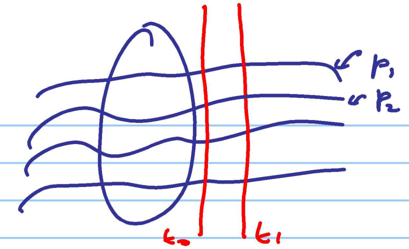
1

Nyquist-Rate Matched filter $\longrightarrow$ ungerboeck

$n(t) \rightarrow \boxed{} \rightarrow \tilde{n}(t)$

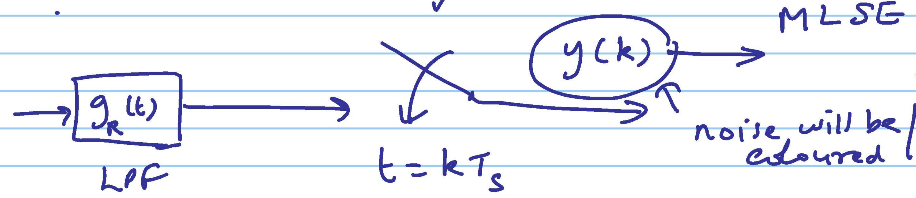
$\frac{1}{W}$

$n(t) \rightarrow$



$n(t_0)$ $n(t_1)$

$E[n_0 n_1] = 0$



$t = kT_s$

noise will be coloured ||

$T_s = T/2, \; \cancel{T/4} \cdot T/8$

$$r(k) = \sum_{l=0}^{L-1} f_l \, I(k-l) + v(k)$$

$I(k), I(k-1) \cdots I(k-L+1)$

$+ q(k)$

$I(k) \in \{+1, -1\}$

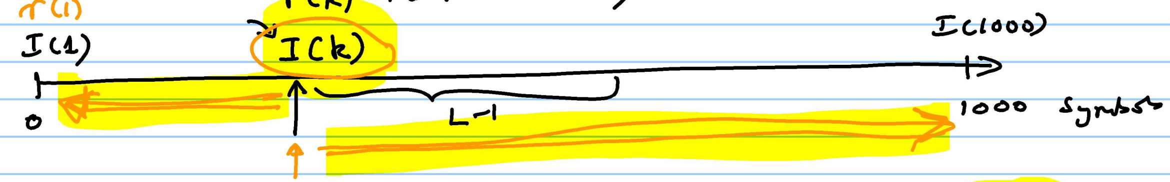
Gaussian white

$L = 2$

$r(k) = f_0 I(k) + f_1 I(k-1) + v(k)$

Optimal Receiver for ISI Channel (1)

$r(k)$ $r(k+1) \ldots r(k+L-1)$

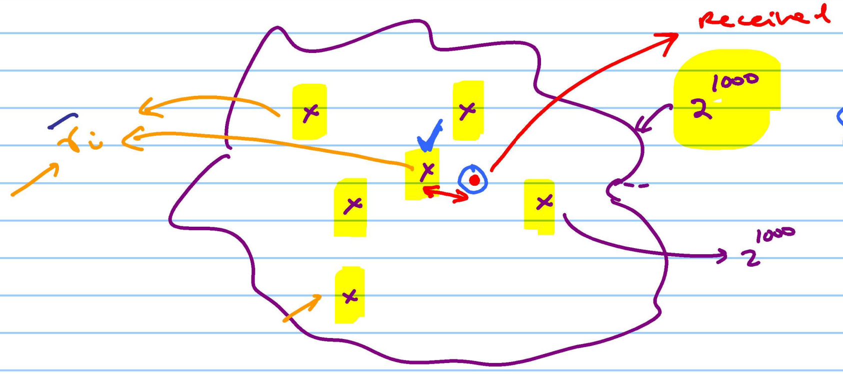


Sequence Estimation $\{I(1), I(2) \ldots\ldots I(k) \ldots\ldots I(1000)\}$

I^{1000}

for $I(k) \in \{+1, -1\}$, how many sequences of length 1000 are there? $\Rightarrow 2^{1000}$

$r^{1000} \longleftrightarrow \{ I_1^{1000}, I_2^{1000} \cdots\cdots I_{2^{1000}}^{1000} \}$ $\quad I_j^{1000}, \; j = 1, 2 \ldots\ldots 2^{1000}$



$r^{1000} = \{r(1), r(2) \cdots\cdots r(1000)\}$

MLSE $\quad |\bar{r} - \bar{I}_j|_2$

$N = 1000$; $M = 2$

$r(1) \cdot r(2) \cdots r(N) \longrightarrow r^N$

Given $k = 1 \ldots\ldots N$ symbols sent

MAP Sequence Estimator $\quad \max_{I^N} p(I^N / r^N)$

$\Rightarrow$ pick $\hat{I}^N = I_i^N$ iff $\boxed{p(I_i^N | r^N) \geq p(I_j^N | r^N)}$ $\forall j \neq i$

$j = 1, 2 \ldots\ldots M^N$

$1, 2 \ldots 2^{1000}$

Bayes'

$$\frac{p(r^N \mid I_i^N)\,\cancel{p(I_i^N)}}{\cancel{p(r^N)}} \leftarrow p(I=1) = p(I=0) = \frac{1}{2}$$

$$p(I_i^N) = \frac{1}{2^{1000}} \;\forall i$$

$$\frac{p(r^N \mid I_j^N)\,\cancel{p(I_j^N)}}{\cancel{p(r^N)}} \qquad p_i = \frac{1}{M^N} \;\forall i$$

$$x^T A x$$

$f_v(\cdot)$

* MLSE

Pick $\hat{I}^N = I_i^N$ iff $p(r^N \mid I_i^N) \geq p(r^N \mid I_j^N), \;\forall j \neq i$

$j = 1 \dots 2^{1000}$

$$\bar{V} = \begin{bmatrix} V(1) \\ V(2) \\ \vdots \\ V(1000) \end{bmatrix}$$

multivariate Gaussian random vector

$$\frac{1}{\sqrt{2\pi \det(R_v)}}\, e^{-\frac{1}{2}(\bar{r} - \bar{r}_i)^T R_v^{-1} (\bar{r} - \bar{r}_i)} \geq \frac{1}{\sqrt{(\;)}}\, e^{-\frac{1}{2}(\bar{r} - \bar{r}_j)^T R_v^{-1} (\bar{r} - \bar{r}_j)}$$

$$R_v = E\left[\bar{V}\bar{V}^T\right]_{N \times N}$$

1000×1000

$-\frac{1}{2}\bar{V}^T R_v^{-1} \bar{V}$

"jointly Gaussian"

$$\rightarrow f_{\bar{V}}(\bar{V}) = \frac{1}{\sqrt{2\pi \det(R_v)}}\, e^{-\frac{1}{2}\bar{V}^T R_v^{-1} \bar{V}}, \quad R_v = E\left[\bar{V}\bar{V}^T\right];$$

$$E[\bar{V}] = \bar{0};$$

joint pdf of a Gaussian random vector

* MLSE for White Gaussian Noise.

$$R_v = E\left[\bar{v}\,\bar{v}^T\right] = \begin{bmatrix} r(0) & r(1) & \cdots & r(1000) \\ r^*(1) & r(0) & & \\ \vdots & & \ddots & \\ r^*(1000) & & & r(0) \end{bmatrix}$$

1000x1000

N x N

r(0) acf

$\xrightarrow{\text{whiteness}}$ $R_v = \begin{bmatrix} r(0) & & 0 \\ & r(0) & \\ 0 & & r(0) \end{bmatrix}$ 1000x1000

$r(0) = \sigma^2$

$\Rightarrow R_v^{-1} = \frac{1}{\sigma^2}\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$

MLSE

→ Pick $\boxed{\hat{I}^N = I_i^N}$ iff

$$e^{-\frac{1}{2}\frac{|r-\hat{r}_i|^2}{\sigma^2}} \geq e^{-\frac{1}{2}\frac{|r-\hat{r}_j|^2}{\sigma^2}}, \quad \forall\, j \neq i,\ j = 1 \ldots 2^{1000}$$

M^N

$I(k) \in \{+1, -1\}$

↳ M-ary signal set

Taking ~~Log~~ on both sides

$$\boxed{|\bar{r}-\hat{r}_i|^2 \leq |\bar{r}-\hat{r}_j|^2} \rightarrow (1a)$$

nearest neighbour Rule

Log Metric

2^{1000} complex. metric

MLSE

$$\sum_{k=1}^{1000} |r(k) - r_i(k)|^2 \leq \sum_{k=1}^{1000} |r(k) - r_j(k)|^2$$

$i = 1 \ldots 2^{1000}$

$j = 1 \ldots 2^{L-1}$

(1b)

$j = 1 \ldots 2^{L-1}$

Viterbi Algorithm for ISI Channels

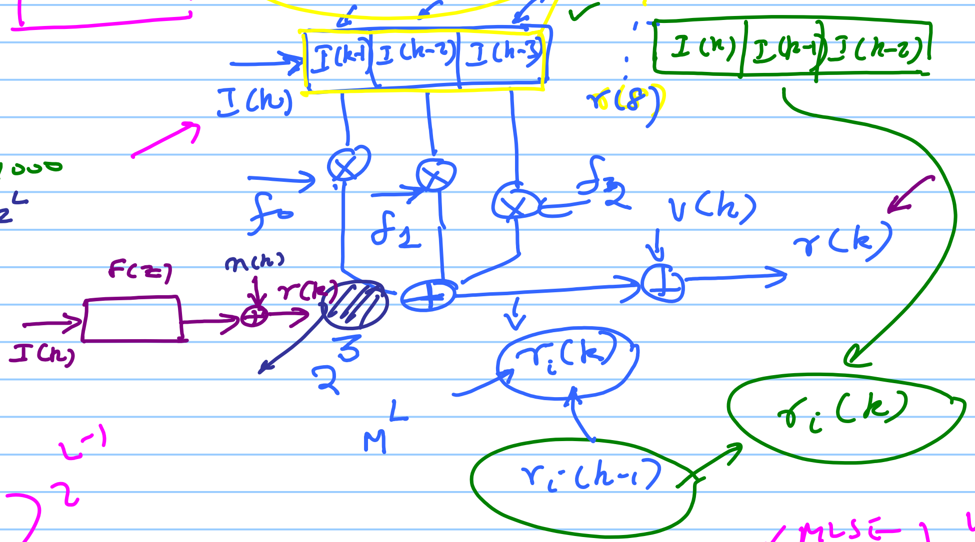
VA $\rightarrow$ MLSE

Training 1000 M^{L-1}

Trellis Diagram :

$L = 3$

$\{ f_0 \; f_1 \; f_2 \}$



$N \rightarrow$ length of the sequence (packet)

$N = 1000$

$L \rightarrow$ channel impulse response (CIR) length

estimated?

$L = 3$

2^{L-1} out of 2^{1000}

2^{L-1}

$2^{3-1} = 2^2 = 4$

$4 \rightarrow$ survivor sequences, each of length 1000

r^{1000}

MLSE — 2^{1000}

VA — ~~4~~

2^2

2^{L-1}

M^{L-1}

64^4

$M = 2$

$L = 3$

r_i 1000

$\begin{bmatrix} r(1) \\ r(3) \\ r(2) \\ r(4) \\ \vdots \\ r(?) \end{bmatrix}$ 2^{1000}

Complexity → Computing complexity, Memory complexity

	MLSE	VA ✓ (L → tap channel)
Computing complexity	2^{1000}	1000×2^{L}
Memory complexity	2^{1000} (M^N)	2^{L-1}; $1000 \times 2^{L-1}$; $5L \times 2^{L-1}$; $\cancel{2L \times 2^{L-1}}$

Recall the 3-tap channel ($L=3$); Assume BPSK Modulation ($M=2$)

$2^{1000} \rightarrow 2^{L-1}$

1000×2^{L}

? → $I(k)$ → $\boxed{I(k-1) \mid I(k-2) \mid I(k-3)}$ → Stage $k-1$ ✓ (f_0, f_1, f_2 ...)

transition

$\boxed{I(k) \mid I(k-1) \mid I(k-2)}$ → stage k ✓

$\hat{f}_0 \longrightarrow f_0$ (estimate)

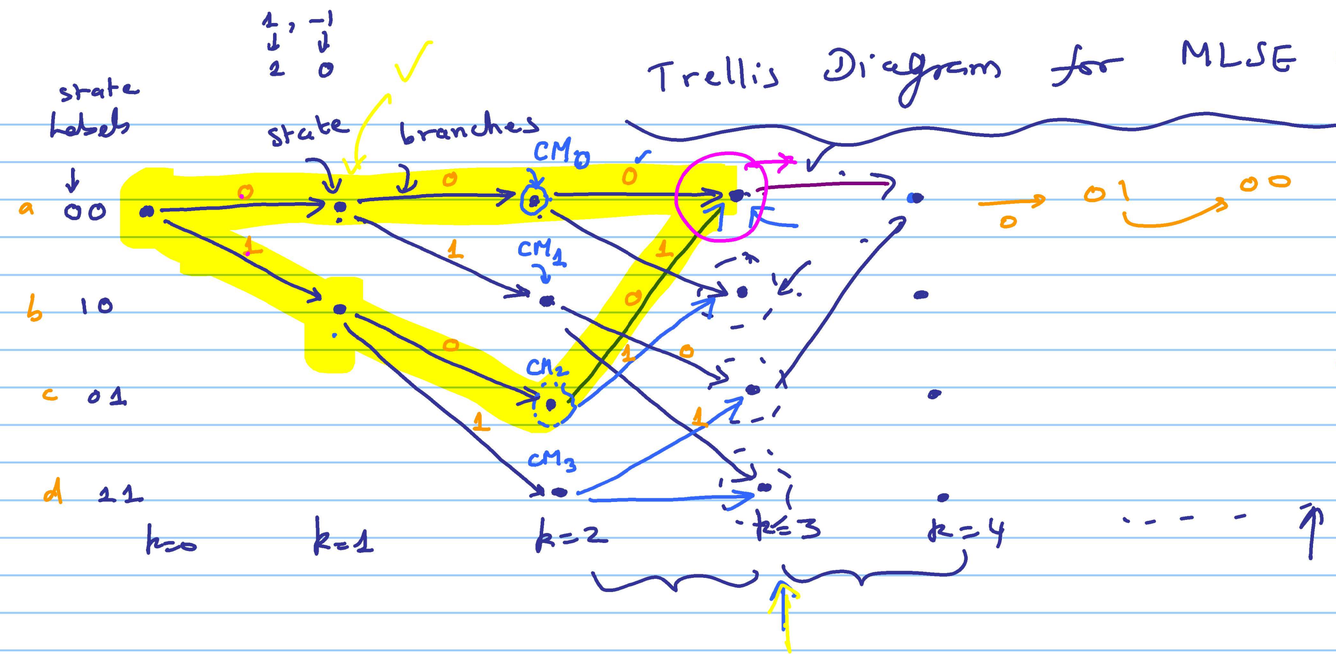
$$r(k) = f_0 I(k) + f_1 I(k-1) + f_2 I(k-2) + n(k)$$

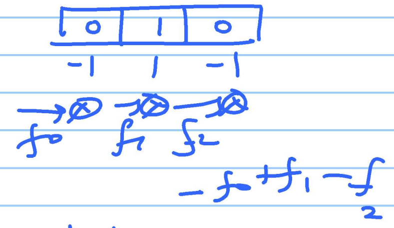
i^{th} hypothesis → $r_i(k) = \hat{f}_0 I_i(k) + \hat{f}_1 I_i(k-1) + \hat{f}_2 I_i(k-2)$

1000×2^{L} hypothesis

$4 = 2^{L-1}$

Trellis Diagram for MLSE using the VA





2 Metrics.

Transition Metric (TM)

Cumulative Metric (CM)

path 000 → d_1

path 100 → d_2

$d_1^{(3)} < d_2^{(3)}$ ✓

$d_1^{(3)} + \alpha < d_2^{(3)} + \alpha$ ✓

$\alpha > 0$

$r(3)$ $r(5)$ — $r_0(5)$

Transition Metric

$$|r(k) - r_i(k)|^2 = TM_i$$

$$CM_0(k-1) + TM_0(k|k-1)$$

$$TM_0(3|2) = |r(3) - r_0(3)|^2$$

$$CM_0(2) + TM_0(3|2) \to ①$$

$$CM_2(\cancel{k}_{k-1}) + TM_2(k|k-1)$$

$$CM_0(k) = CM_0(k-1) + TM_0(k|k-1)$$

$$CM_2(k) = CM_2(k-1) + TM_2(k|k-1)$$

Read #5

$$|\bar{V}_1| < |\bar{V}_2|$$

$$\not\Rightarrow |\bar{V}_1 + \bar{\varphi}| < |\bar{V}_2 + \bar{\varphi}|$$

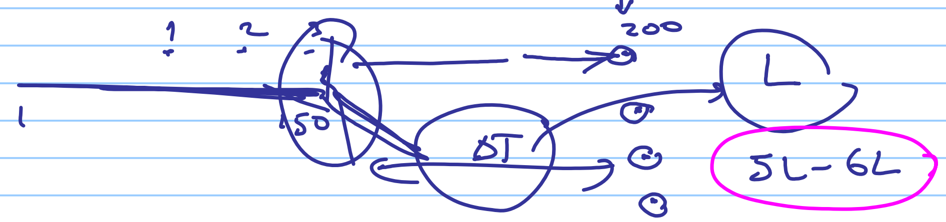
S. Haykin

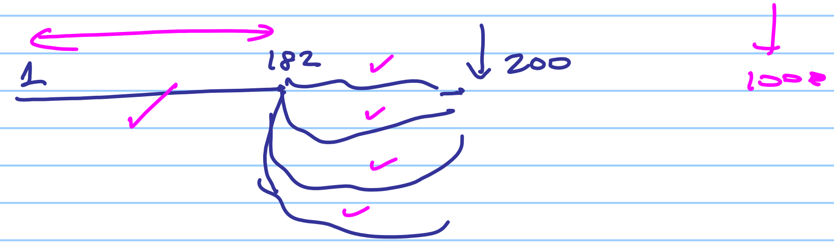
Proakis

if $CM_0(k) < CM_2(k)$

winner.

$$CM_0(k) + \varepsilon^{k} < CM_2(k) + \varepsilon^{k}$$





Viterbi Algorithm for ISI Channels

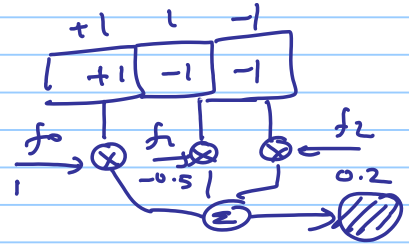
Given a channel $\boxed{F(z) = 1 - 0.5z^{-1} + 0.2z^{-2}}$, and data symbols $I(k) \in \{+1, -1\}$ the receiver measurements $r(k) = \sum_{l=0}^{2} f_l I(k-l) + n(k)$ over the first 6 symbol intervals is given by

k	1	2	3	4	5	6
r(k)	1.1	0.6	−1.5	1.6	−1.5	−0.2

Set up the trellis for the VA, and determine (a) the smallest Cumulative Metric (CM) at time k=6 & (b) the corresponding survivor sequence

* How many states? $\Rightarrow$ $M=2$, $L=3$ $\quad 2^{3-1} = 4$ states

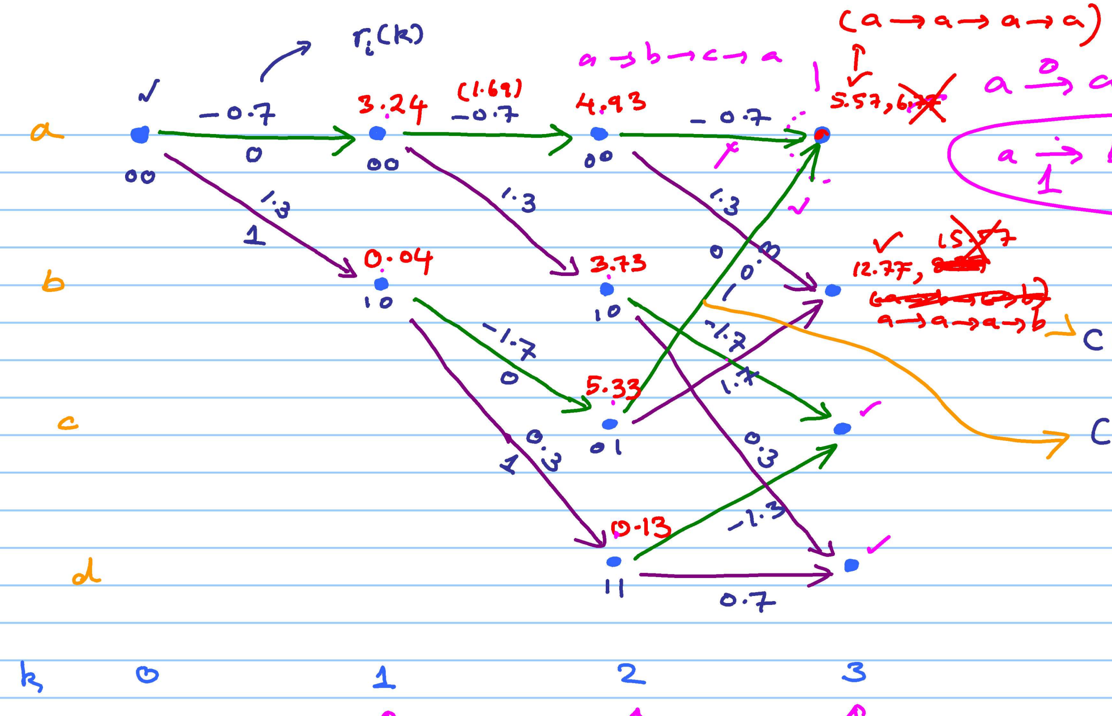
$-1 + 0.5 - 0.2 = -0.7$



$1 + 0.5 - 0.2 = 1.3$

* "0" $\rightarrow$ −1

"1" $\rightarrow$ 1



(a→a→a→a)

$a \xrightarrow{0} a \xrightarrow{0} a \xrightarrow{0} a$

$a \xrightarrow[1]{} b \xrightarrow[0]{} c \xrightarrow[0]{} a$

~~15.57~~

$$CM_\alpha^{-1}(k) = CM_a^{-1}(k-1) + TM_{a\to a}(k|k-1)$$

$$CM_\beta^{-1}(k) = CM_c^{-1}(k-1) + TM_{c\to a}(k|k-1)$$

$$CM_a^{-1}(k) \longrightarrow \min\left\{ CM_\alpha^{-1}(k),\ CM_\beta^{-1}(k) \right\}$$

Abend & Fritchman 1964 | Forney 1972

$r(1) = 1.1$

$(r - r_i)^2$

$$CM_a^{-1}(1) = CM_a^{-1}(0) + TM_{a\to a}(1|0) \qquad (1.1 - (-0.7))^2 = 1.8^2$$

$$= 0 + 3.24$$

$$CM_b^{1}(1) = CM_a^{-1}(0) + TM_{a\to b}(1|0) \qquad (1.1 - 1.3)^2 = 0.04$$

$r(2) = 0.6$

$(0.6 - (-0.7))^2$

Sub-optimal MLSE : M-algo ; k-algo ; RSSE ; DF with VA

M^{L-1} | Fano Algorithm
| Stack Algorithm