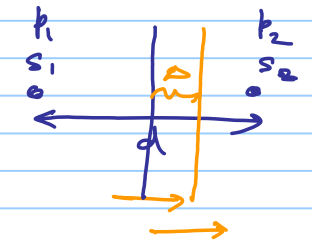
Lesson 2b: Pe for other signals, Union Bound, Chernoff Bound

Note Title 9/26/2017

Exercise

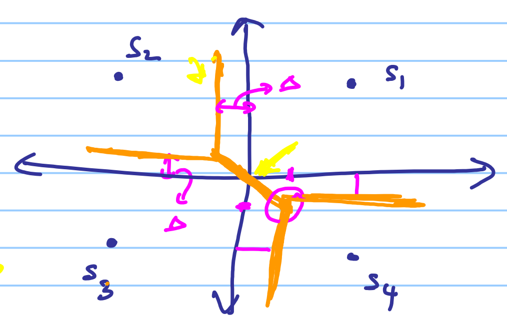
show that

$$\boxed{\Delta = \frac{N_0/2}{d} \ln\left(\frac{p_1}{p_2}\right)}$$



Example :

QPSK



$p(s_1) = p(s_3) = \frac{1}{3}$

$p(s_2) = p(s_4) = \frac{1}{6}$

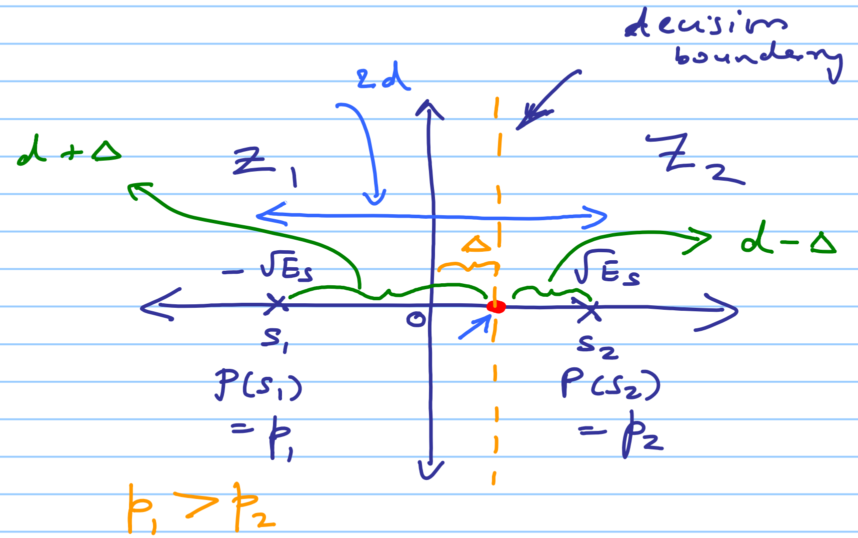
Chernoff Bound

$$P_E = p_1 \cdot \underbrace{\frac{1}{2}\,\mathrm{erfc}\left(\frac{\frac{d}{2}+\Delta}{\sqrt{N_0}}\right)}_{q_1} + p_2\, \underbrace{\frac{1}{2}\,\mathrm{erfc}\left(\frac{d/2-\Delta}{\sqrt{N_0}}\right)}_{q_2}$$

Lesson 2b

P_e, Union-Bound, Chernoff Bound

(*) Revisit finding P_E for unequal priors



$$r(t) = s_i(t) + n(t)$$
$$\Downarrow$$
$$r = s_i + n$$

rv

At the decision boundary, MAP metrics are equal

$$p(r \in Z_1 | s_1)\, p_1 = p(r \in Z_2 | s_2)\, p_2$$

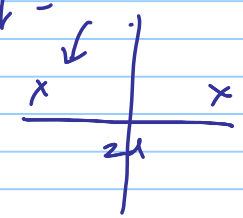
→ the locus is $\perp^r$ to the line $s_1 \longleftrightarrow s_2$ but not a bisector if $p_1 \neq p_2$

$$\frac{1}{\sqrt{\pi N_0}} e^{-\frac{(d+\Delta)^2}{N_0}} p_1 = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(d-\Delta)^2}{N_0}} p_2$$

Prob. that the noise rv takes a value $d+\Delta$ $(\sqrt{E_s}+\Delta)$

rv takes a value $d-\Delta$ $(-\sqrt{E_s}+\Delta)$

$q \overset{\Delta}{=} \frac{1}{2}\,\mathrm{erfc}\left(\frac{d}{\sqrt{N_0}}\right)$



$$\Delta = \frac{N_0/2}{2d} \ln\left(\frac{p_1}{p_2}\right), \quad p_1 > p_2$$

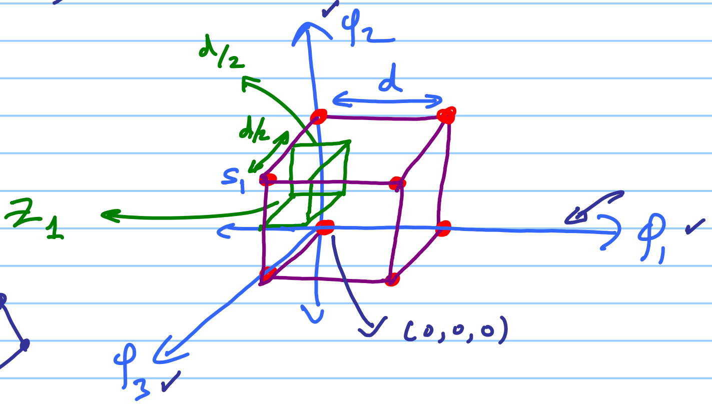
$\therefore$ the avg. prob. of symbol error (bit)

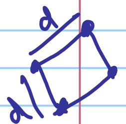
$$P_E = p_1 \cdot \underbrace{\frac{1}{2}\,\mathrm{erfc}\left(\frac{d+\Delta}{\sqrt{N_0}}\right)}_{\cancel{q_1}} + p_2 \cdot \underbrace{\frac{1}{2}\,\mathrm{erfc}\left(\frac{d-\Delta}{\sqrt{N_0}}\right)}_{\cancel{q_2}}$$

(*) P_E for "Vertices of a Hypercube" $\longrightarrow$ signal waveforms generated from binary codes

$\boxed{M = 2^N}$ $\quad N \rightarrow$ # of dimensions (# of basis fn.)

$\rightarrow N = 3 \Rightarrow M = 2^3 = 8$;





$$\bar{r} = \bar{s}_i + \bar{n} \leftarrow \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$p\left(\bar{r} \in Z_1 \,\middle|\, \bar{s}_1 \text{ sent}\right) = p\left(-\infty < n_1 < d/2,\ -\infty < n_2 < d/2,\ -\infty < n_3 < d/2\right)$$

joint pdf

Since noise is (assumed to be) i.i.d

$$= p\left[-\infty < n < d/2\right]^3$$

$$p[-\infty < n < d/2] = \int_{-\infty}^{d/2} f_N(\alpha)\, d\alpha = 1 - \underbrace{\int_{d/2}^{\infty} f_N(\alpha)\, d\alpha}_{q}$$

$$= 1 - q$$

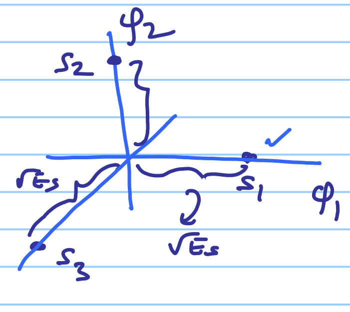
$\therefore P_c = (1-q)^3$

$$\Rightarrow \boxed{P_E = 1 - (1-q)^3}$$

Exercise : For the minimum energy constellation (i.e., origin at "0") show that $E_a = E_i = N\left(d^2/4\right)$; for $N=3$, $3\,d^2/4$;

Exercise : $\begin{cases} \text{What is the } P_E \text{ for arbitrary } N \text{ ?} & 1-(1-q)^N \\ \text{what is } d \text{ for arbitrary } N, \text{ in terms of } N \text{ and } E_a \text{ ?} \end{cases}$

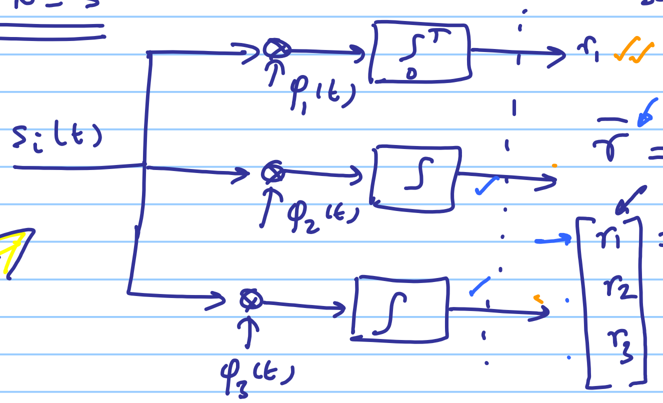
(*) M-ary orthogonal Signals



Here, $N = M$ $s_i(t) = \sqrt{E_s}\, \phi_i(t)$, $i = 1 \dots M$ (N)

N=3

Say $\bar{s}_1$ was sent $r(t) = s_1(t) + n(t)$



$\bar{r} = \bar{s}_1 + \bar{n}$ $\longrightarrow$ $r_1 = \sqrt{E_s} + n_1$

$\nu - \sqrt{E_s}$

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \sqrt{E_s} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

$p(B) = \sum p(B/A)\, p(A)$

$f_N(\alpha) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{\alpha^2}{N_0}}$

using Bayes rule

$$p(\bar{r} \in Z_1 | s_1) = \int_{-\infty}^{+\infty} \underbrace{p(\bar{r} \in Z_1 | s_1, r_1 = \nu)}_{= \, p[n_2 < \nu,\ n_3 < \nu]} \cdot \underbrace{p[r_1 = \nu]}_{f_{r_1}(\nu) = f_{n_1}(\nu - \sqrt{E_s})} d\nu$$

$$\left[1 - \int_{\nu}^{\infty} f_N(\alpha)\, d\alpha\right]^2$$

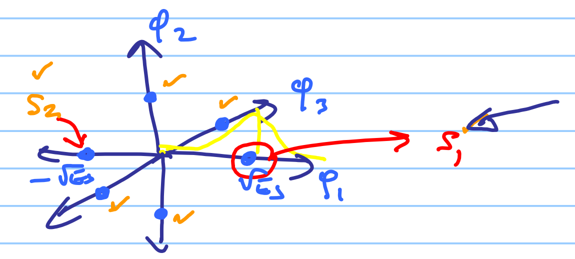
$$P_c = P\left[\bar{r} \in Z_1 \middle| s_1\right] = \int_{-\infty}^{+\infty} f_N(\nu - \sqrt{E_s}) \cdot \left(1 - \underbrace{\int_{\nu}^{\infty} f_N(\alpha)\, d\alpha}\right)^{2} d\nu$$

$M-1$ (or $N-1$)

$$P_E = 1 - P_c$$

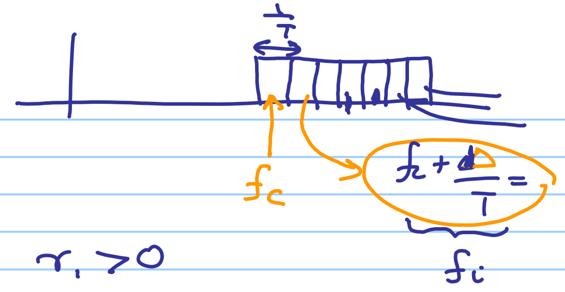
(*) <u>Exercise</u> : "Simplex signals" are obtained from orthogonal signals by translating the "centroid" to the origin. For $N=3$, find the E_{avg} for the simplex case.

(*) <u>Exercise</u> : Bi- orthogonal signals $\boxed{M = 2N}$



$$\begin{cases} \rightarrow s_i(t) = A \cos\left(\overbrace{2\pi f_c t + 2\pi f_i t}^{2\pi(f_c + i\Delta f)t}\right) \\ \rightarrow s_i'(t) = -A \cos(2\pi f_c t + 2\pi f_i t) \end{cases}$$

d, $-d$



Let $s_1(t)$ be sent

$$P(\bar{r} \in Z_1 \mid s_1 \text{ sent}) = \int_{-\infty}^{\infty} \underbrace{P(\bar{r} \in Z_1 \mid s_1, \overbrace{r_1 = \nu > 0}^{r_1 > 0})}_{= P[-\nu < n_2 < \nu,\ -\nu < n_3 < \nu] = \left[\int_{-\nu}^{\nu} f_N(\alpha)\, d\alpha\right]^2} f_r(r_1 = \nu > 0)\, d\nu$$

&

$$f_r(r_1 = \nu > 0) = \begin{cases} f_N(r_1 - \sqrt{E_s}), & r_1 - \sqrt{E_s} > 0 \\ 0, & \text{otherwise} \end{cases}$$

$r_1 = \sqrt{E_s} + n_1$

$r_1 > 0$

$\Rightarrow \sqrt{E_s} + n_1 > 0$

$\Rightarrow n_1 > -\sqrt{E_s}$

$1 - 2\int_{\nu}^{\infty} f_N(\alpha)$

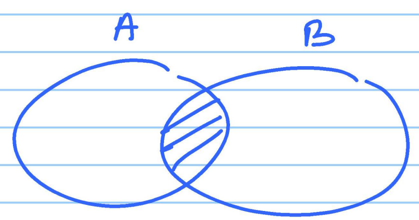
$$\therefore\ P_c = \int_{-\sqrt{E_s}}^{\infty} f_N(\nu - \sqrt{E_s}) \left[1 - 2\int_{\nu}^{\infty} f_N(\alpha)\, d\alpha\right]^{2} d\nu \ ; \quad P_E = 1 - P_c \ ;$$

$N-1$

#

Union-Bound on P_E

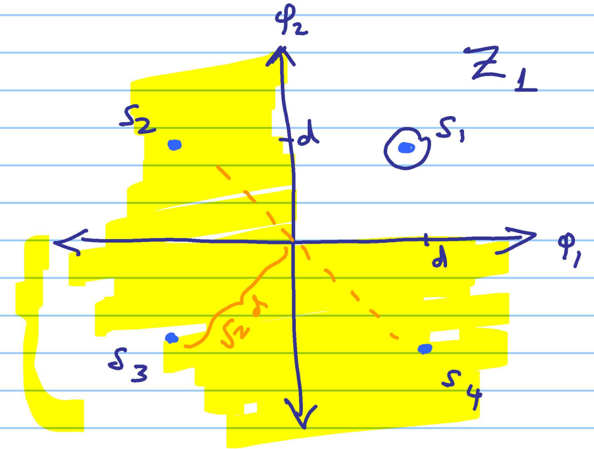
$P(A \cup B \cup C \cup D) \leq P_2(A-B) + P_2(A-C) +$



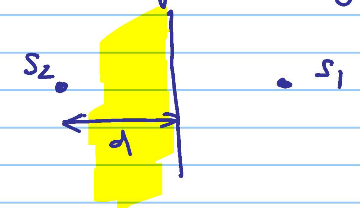
$P(A \cup B) \leq P(A) + P(B) - \cancel{P(A \cap B)}$

eg : QPSK

$P(s_i) = \frac{1}{4}$;



Pair-wise error probability



✓ (*) $P_{12} = \frac{1}{2} erfc\left(\frac{d}{\sqrt{N_0}}\right)$ ✓

✓ * s_1 & s_4 $\Rightarrow$ $P_{14} = \frac{1}{2} erfc\left(\frac{d}{\sqrt{N_0}}\right)$

✗ * s_1 & s_3 $\Rightarrow$ $P_{13} = \frac{1}{2} erfc\left(\frac{\sqrt{2}d}{\sqrt{N_0}}\right)$

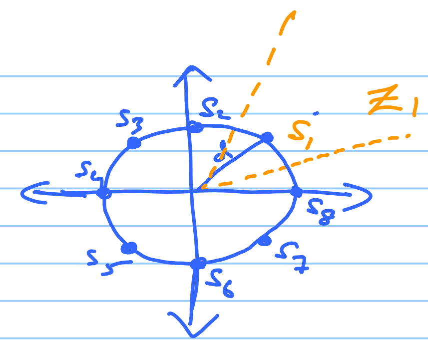
$q' < q$

$P_E \leq \underbrace{\overset{q}{P_{12}} + \overset{q}{P_{14}}}_{2q} + \overset{q'}{P_{13}}$

$2q + \cancel{q'}$

Recall, for QPSK, the accurate P_E

$$P_E = 2q - \cancel{q^2}$$



Exercise: For the 8-PSK constellation, use union bound

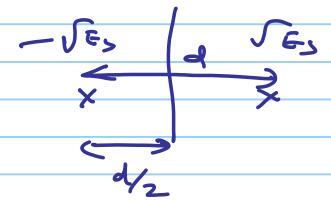
(a) The full union bound. ✓

(b) only on the Nearest Neighbours ✓ $(s_1 - s_2 \ \& \ s_1 - s_8)$ ✓

(c) wrt to also $(s_1 - s_3)$ & $(s_1 - s_7)$

Compare them with the true P_E

$d = 2\sqrt{E_s}$



$\boxed{r = \alpha s_i + n}$

r.v.

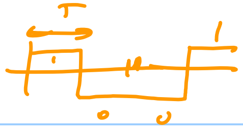
CHERNOFF BOUND

$$\boxed{\frac{1}{2}\,\mathrm{erfc}\left(\frac{\alpha}{\sqrt{2}}\right) < e^{-\alpha^2/2}}$$

$$\underbrace{\frac{1}{2}\,\mathrm{erfc}\left(\frac{d/2}{\sqrt{N_0}}\right)}_{q} = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\frac{d}{\sqrt{2N_0}}}{\sqrt{2}}\right) < e^{-d^2/4N_0}$$

$$\boxed{q < e^{-E_s/N_0}}$$

P_b for "Flat Fading" Rayleigh Channel



$T \,\&\, \tau_D$

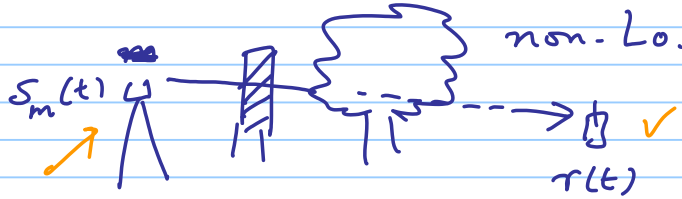
$\frac{\tau_D}{T}$

τ_D delay spread $\tau_D = \tau_r - \tau_o$

$\alpha_1 < \alpha_o$

$\tau_1 > \tau_o$

LoS → Line of sight



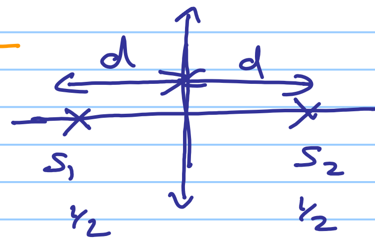
non-LoS →

wireless link → Path Loss

→ Shadow Loss (Long-term Fading)

→ Short-term Fading $\begin{cases} \text{LoS} \\ \text{non-LoS} \end{cases}$

Rayleigh pdf



$P_E = \frac{1}{2}\,\text{erfc}\left(\frac{d}{\sqrt{N_o}}\right)$

$r(t) = s_m(t) + n(t)$ ← noise

$r(t) = 3.5\, s_m(t) + n(t) \implies P_E = \frac{1}{2}\,\text{erfc}\left(\frac{3.5\,d}{\sqrt{N_o}}\right)$; $\sqrt{\text{SNR}}$

$\alpha(t = t_o) = \alpha(t_o)$ r.v.

→ $r(t) = c(t)\, s_m(t) + n(t)$ $\quad \alpha(t) \to$ Fade Random Process / Fade Random Variable

pp. 627 – 631
Proakis & Salehi

Let the channel gain

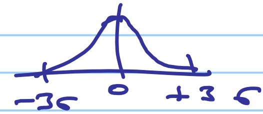
$\Rightarrow c(t) = c_I(t) + j\, c_Q(t)$

$= \alpha(t)\, e^{j\phi(t)}$

$c_I(t)$ & $c_Q(t) \rightarrow$ Gaussian Random Process

$c_I(t_0)$, $c_Q(t_0)$

$\mathcal{N}(0, \sigma^2)$



$\alpha(t) = \sqrt{c_I^2(t) + c_Q^2(t)}$ & $\phi(t) = \tan^{-1} \dfrac{c_Q(t)}{c_I(t)}$

Envelope

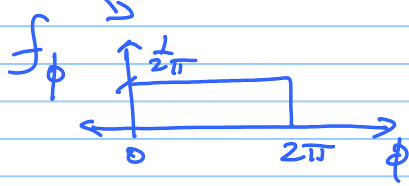
Phase

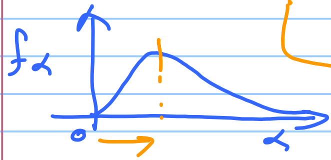
$\alpha(t_0)$

Rayleigh pdf

$$\boxed{f_\alpha = \begin{cases} \dfrac{\alpha}{\sigma^2} e^{-\alpha^2/2\sigma^2}, & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases}}$$

1





$c(t) \simeq c = \boxed{\alpha\, e^{j\phi}}$

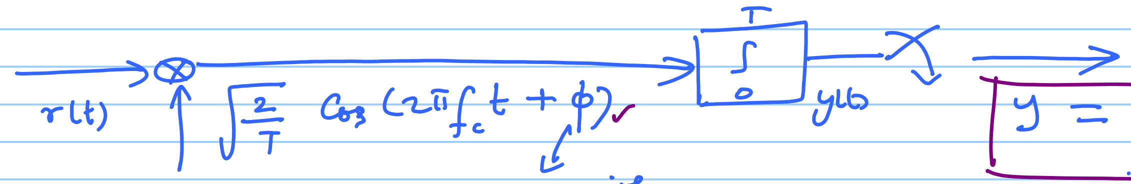
Slowly fading, frequency non-selective model

Let BPSK (Bipolar PAM) signals be used

$s_m(t) = \sqrt{\dfrac{2E_b}{T}} \cos\left(2\pi f_c t + m\pi\right),$ $m = 0$ or 1

$$r(t) = \alpha \sqrt{\frac{2E_b}{T}} \cos\left(2\pi f_c t + m\pi + \phi\right) + n(t), \quad m = 0 \text{ or } 1$$

$\alpha \to$ Rayleigh r.v.



assuming this "known" by the Rx (estimated)

$$\boxed{y = \alpha\sqrt{E_b} \cos m\pi + n} \qquad \alpha \geq 0$$

$\alpha \to$ fixed (constant)

$$P_2(\alpha) = \frac{1}{2} \operatorname{erfc}\left(\frac{\alpha\sqrt{E_b}}{\sqrt{N_0}}\right)$$

P_e

$$= \frac{1}{2} \operatorname{erfc}\left(\alpha\sqrt{\frac{E_b}{N_0}}\right); \quad \text{②}$$

$\alpha \to$ Rayleigh

Average P_e (over the distribution of α)

$$P_e = \int_0^\infty P_2(\alpha)\, f_\alpha \, d\alpha$$

Expectation $E[x] = \int x\, f_X(x)\, dx$

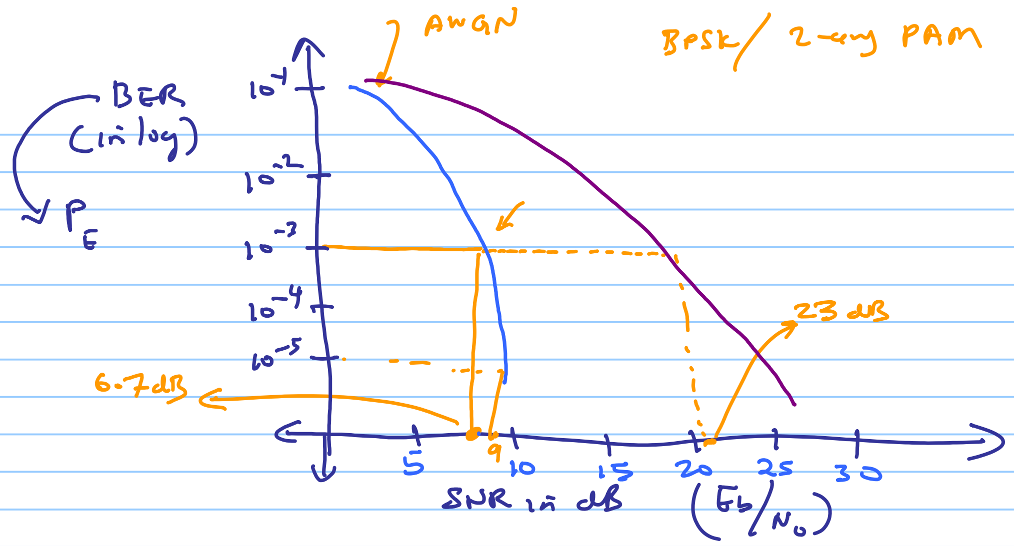
Putting ① & ② into this integral

$$\bar{\rho}_b = \left(\frac{E_b}{N_0}\right) \cdot \underbrace{E(\alpha^2)}_{2\sigma^2}$$

average received SNR

$$P_e = \frac{1}{2}\left[1 - \sqrt{\frac{\bar{\rho}_b}{1 + \bar{\rho}_b}}\right]$$

BER Curve



BPSK/ 2-ary PAM

10^{-5} BER → 9 dB SNR for AWGN
→ 30 dB SNR for Rayleigh ✓

$10 \log_{10} (10)^2$

#

40 μsec ANSI IS-136 / 2G

$\approx$ 2-5 μsec ~~ANSI-95~~ IS-95