Lesson 2a -- Optimum Detection Principle – MAP, ML, and NN rules -- Decision Regions -- Computing the Probability of Symbol Error

Note Title 8/8/2017

$\alpha\,\delta(t) = h(t)$

$s(t) \rightsquigarrow$ $r(t)$

$|\alpha| < 1$ $n(t)$ $\frac{1}{\alpha}$

Optimum Detection → $\boxed{r(t) = s_m(t) + n(t)}$ (A)

Additive White Gaussian Noise (AWGN) Channel → ?

$\{\phi(t)\}$ $m = 1 \ldots M$ sufficient statistics

$\boxed{r = s_m + n}$ (B) $s_m \in \left\{ +d\sqrt{\frac{E_g}{2}}, -d\sqrt{\frac{E_g}{2}} \right\}$

$s_m \in \{+d, -d\}$

Probability of Correct Decision

exclusive

$$P(c) = \sum_{m=1}^{2} P(c \mid s_m)\, P(s_m) \quad P\left(\begin{matrix}\text{Correct}\\\text{decision}\end{matrix} \,\middle|\, \begin{matrix}s_m\ \text{was}\\\text{transmitted}\end{matrix}\right)$$

$$= \sum_{m=1}^{2} p(s_m) \int_{R_m} p(r \mid s_m)\, dr \qquad P(r \in R_1)$$

R_1 R_2 s_1 s_2 $-d$ $+d$ ϕ

Th → threshold

Decision Boundary ?

$$P(c) = \sum_{m=1}^{2} \int_{R_m} p(r \mid s_m)\, p(s_m)\, dr$$

$p(a|b)\,p(b) = p(b|a)\,p(a)$

To maximise $P(C)$ $\Rightarrow$ maximise $p(r|s_m)\cdot p(s_m)$ $\forall s_m$

$\Rightarrow$ maximise $p(s_m|r)\,p(r)$ $\forall s_m$

$\Rightarrow$ maximise $p(s_m|r)$ $\forall s_m$ since $p(r)$ is common

MAP — probability that the symbol s_m was sent having "observed" r

Maximum A Posteriori Probability

Maximum A Posteriori (MAP) Rule

Pick $\boxed{\hat{S} = s_i}$ $(i = 1, 2 \ldots M)$ iff $\boxed{p(s_i|r) \geq p(s_j|r)}$ $\forall j \neq i$ $\longrightarrow$ (1)

Symbol decision

$$= \left(\Rightarrow \frac{p(r|s_i)\,\cancel{p(s_i)}}{\cancel{p(r)}} \geq \frac{p(r|s_j)\,\cancel{p(s_j)}}{\cancel{p(r)}} \right)$$ Bayes' Rule

and if $p(s_i) = \frac{1}{M}$ $\forall i$ $\longrightarrow$ equal priors / symbol probabilities

Maximum Likelihood (ML) Rule

Pick $\hat{s} = s_i$ iff $\boxed{p(r|s_i) \geq p(r|s_j)}$, $\forall j \neq i$ → (2)

Recall $r = s_i + n$ → Gaussian pdf → $f_N(n) = \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} e^{-n^2 / 2 \times \frac{N_0}{2}}$ ✓

$\boxed{\sigma^2 = \frac{N_0}{2}}$

$$p(r|s_i) = P(\text{Noise takes a value } r - s_i)$$

$$= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-s_i)^2}{N_0}}$$ ✓

Pick $\hat{s} = s_i$ iff $\cancel{\frac{1}{\sqrt{\pi N_0}}} e^{-\frac{(r-s_i)^2}{N_0}} \geq \cancel{\frac{1}{\sqrt{\pi N_0}}} e^{-\frac{(r-s_j)^2}{N_0}}$ $\forall j \neq i$ & $j = 1 \dots M$

exp(·) → Monotic fn.

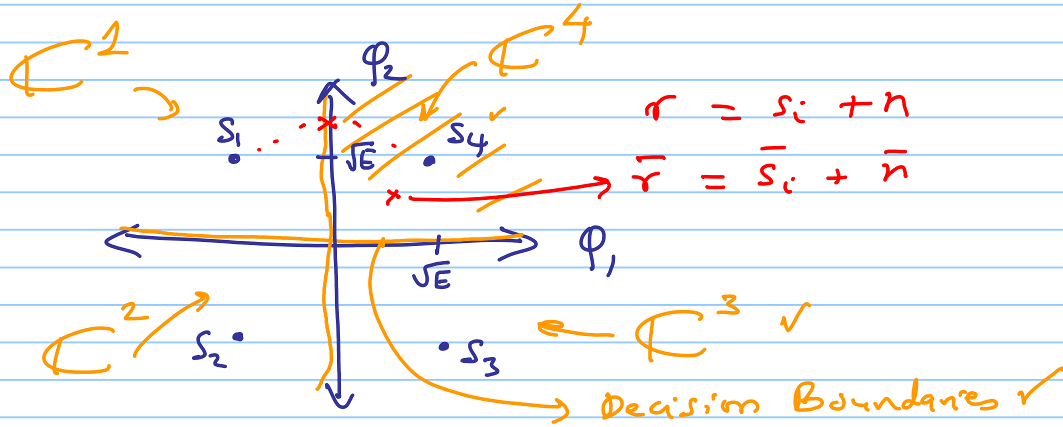
Nearest Neighbour

Taking $\log_e(\cdot)$ on both sides and recognising the $(-)^{ve}$ sign

Pick $\hat{s} = s_i$ iff $(r - s_i)^2 \leq (r - s_j)^2 \quad \forall\, j \neq i$ & $j = 1 \dots M$

Decision Boundaries & Decision Regions

eg: QPSK



$r = s_i + n$

$\bar{r} = \bar{s}_i + \bar{n}$

$\mathbb{R} \rightarrow$ Real

$\mathbb{C} \rightarrow$ Complex

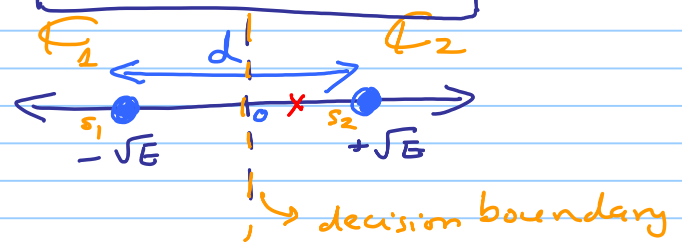
→ Probability of Symbol Error (SEP)

$$\boxed{r = s_i + n}$$

$s_i \in \{+\sqrt{E_s}, -\sqrt{E_s}\}$ $p(s_1) = p(s_2) = \frac{1}{2}$ ✓

Noise $n \in \mathcal{N}\left(0, \frac{N_0}{2}\right)$ ✓

$E[n] = 0$; $E[n^2] = \frac{N_0}{2} = \sigma_n^2$



Let s_1 be sent $\boxed{r = -\sqrt{E_s} + n}$ ←

decision error

$$p(e) = \underbrace{p(e|s_1)}_{P_1}\, p(s_1) + p(e|s_2)\, p(s_2)$$

$$P_1 = P(r \in C_2 \mid s_1 \text{ sent}) = \int_{C_2} p(r|s_1)\, dr$$

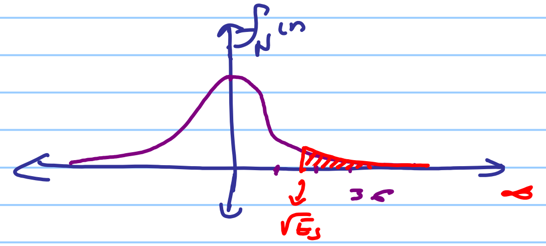
$$= P(r \geq 0 \mid s_1 \text{ sent})$$

$$= P(n \geq \sqrt{E_s} \mid s_1 \text{ sent})$$

~~s_2 sent~~

$= P(n \geq \sqrt{E_s})$ ← n & s_i are statistically independent

$$= \int_{\sqrt{E_s}}^{\infty} f_N(n)\,dn = \int_{\sqrt{E_s}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-n^2/N_0}\,dn \rightarrow (3)$$



use $\boxed{\beta \triangleq \frac{n}{\sqrt{N_0}}}$ $\Rightarrow d\beta = \frac{dn}{\sqrt{N_0}}$ & when $n = \sqrt{E_s}$, $\beta = \sqrt{\frac{E_s}{N_0}}$

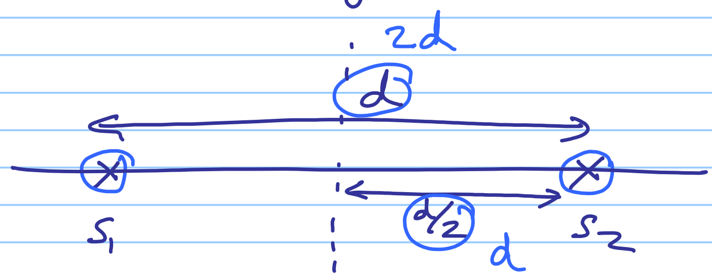
$$P_1 = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_s/N_0}}^{\infty} e^{-\beta^2}\,d\beta \rightarrow (4)$$

$$= \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_s}{N_0}}\right); \checkmark$$

$$= Q\left(\sqrt{\frac{E_s}{N_0/2}}\right); \checkmark$$

$$= q$$

In general



decision boundary ($\perp^r$ bisectors connecting the 2 points)

$$P_1 = \frac{1}{2}\,\mathrm{erfc}\left(\frac{d/2}{\sqrt{N_0}}\right) = Q\left(\frac{d/2}{\sqrt{N_0/2}}\right) \equiv q$$

d

Ex #1:

$$\boxed{\bar{r} = \bar{s}_i + \bar{n}}$$

complex $\longrightarrow$ $r = s_i + n$

$s_{iI} + j s_{iQ}$

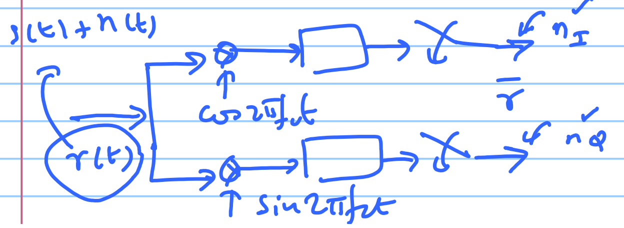
$\mathcal{N}(0, 0; \frac{N_0}{2}, \frac{N_0}{2})$ (i, q, i, q)

$n_I + j n_Q$

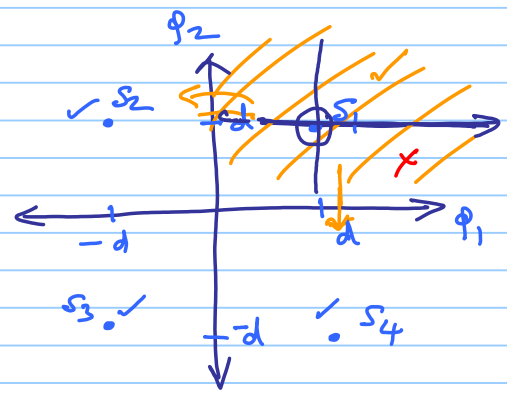
iid

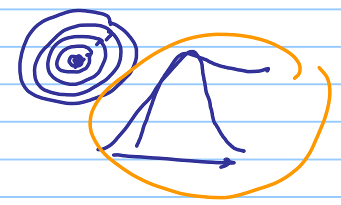
uncorrelated $\longrightarrow n_I \perp^r n_Q$

$\longrightarrow E[n_I n_Q] = 0$



QPSK





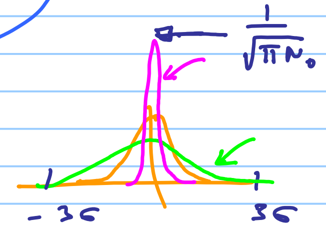
QPSK signal sent thr an AWGN channel

$P_e \longrightarrow$ Avg. Probability of Symbol Error

$P_c \longrightarrow$ Avg. " " Correct Symbol Detection

$$P_e = 1 - P_c$$

$$\frac{1}{\sqrt{\pi N_0}} e^{-\frac{n}{N_0}}$$

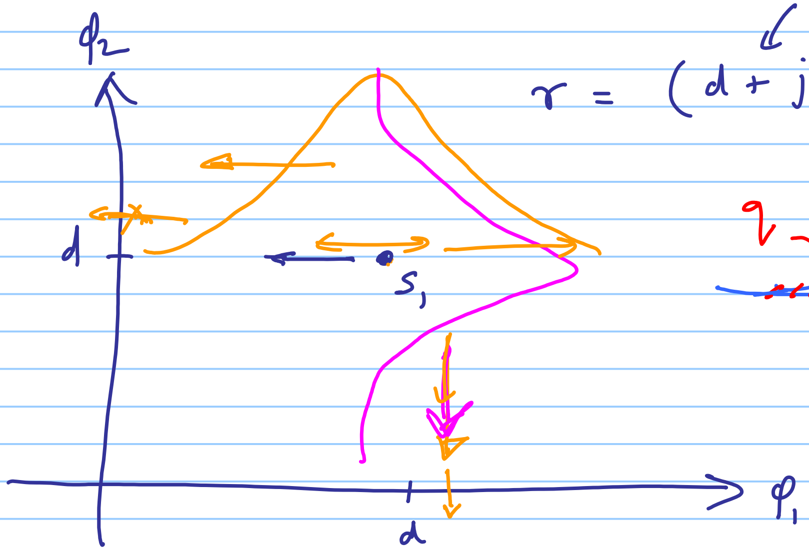


$$r = \underbrace{(d + jd)}_{s_1} + \underbrace{(n_I + j n_Q)}_{n}$$

$$\boxed{r = s_i + n}$$

$\theta = 0$

$$r = e^{j\theta} s_i + n$$



$$\therefore\ P(c \mid s_1) = (1-q)\cdot(1-q)$$

s_2, s_3, s_4 and

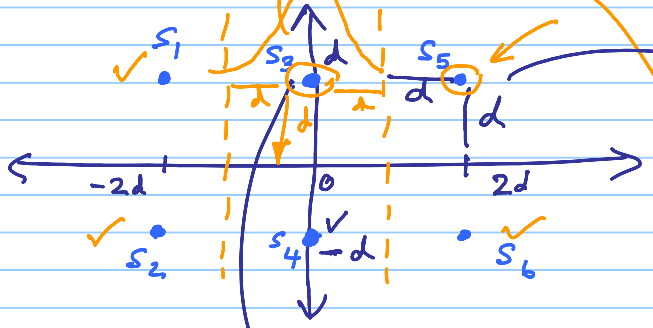
$$P(c) = \sum_{i=1}^{4} P(c \mid s_i)\, P(s_i) \qquad P(s_i) = \frac{1}{4}$$

$= P(c)$

$$P_e = 1 - P_c = 1 - (1-q)(1-q) = 2q - q^2$$

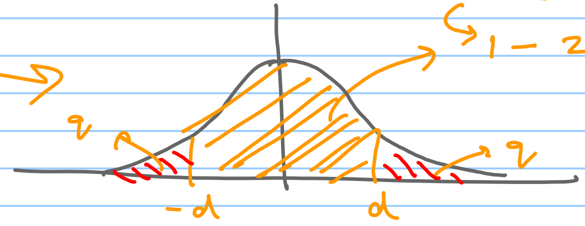
$$\boxed{q = \frac{1}{2}\,\mathrm{erfc}\left(\frac{d}{\sqrt{N_0}}\right)}$$

Ex #2



$\cancel{P_c} = (1-q)(1-q)$

$P(C|s_5)$



$n_I \to 1-2q$

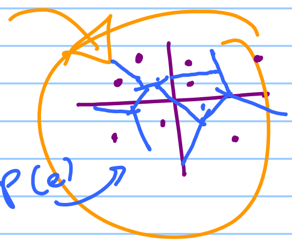
$n_Q \to (1-q)$

$$P(C|s_3) = (1-2q)(1-q)$$

$$P(C) = \frac{4}{6}(1-q)^2 + \frac{2}{6}(1-2q)(1-q)$$

$$P(e) = 1 - P(C)$$

$P(s_i) \neq \frac{1}{M} \quad \forall i \qquad s_1, s_2, \dots, s_M$



$P(e)$

Voronoi Diagram

$$\boxed{r = s_i + n}$$

$\mathcal{N}(0, \frac{N_0}{2})$