Car-following models: A delayed dynamical view of transportation networks

Gopal Krishna Kamath M

Research advisors: Dr. Krishna Jagannathan and Dr. Gaurav Raina
PhD Seminar 1

Department of Electrical Engineering
Indian Institute of Technology Madras

August 7, 2017
(1) *Microscopic:* Individual element (vehicle) dynamics
Modeling transportation networks

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(2) **Mesoscopic**: Small groups of homogeneous elements (vehicles)
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(3) *Macroscopic*: Evolution of aggregate quantities; flow, density, etc
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(3) **Macroscopic**: Evolution of aggregate quantities; flow, density, etc
Car-following models

Class of dynamical models
- Temporal variation of acceleration, velocity, position
- Mimic human drivers’ decisions
- Circular loop/infinite highway
Car-following models

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- Basic philosophy
  - Synchronize velocity with vehicle directly ahead
Car-following models

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Basic philosophy
- Synchronize velocity with vehicle directly ahead

Examples

- Classical car-following model[1]
- Optimal velocity model[2]
- Intelligent driver model[3]

Two variants of car-following models

Circular loop
Two variants of car-following models

Circular loop

Infinite highway
Overview

Problems statement

Insight into control laws that yield desirable traffic flow
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Insight into control laws that yield desirable traffic flow

Contributions
- Phenomenological insight into “phantom jams”
- Ensure lack of jerky vehicular motion (non-oscillatory convergence)
- Characterize time taken by platoon to equilibrate
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Insight into control laws that yield desirable traffic flow

Contributions
- Phenomenological insight into “phantom jams”
- Ensure lack of jerky vehicular motion (non-oscillatory convergence)
- Characterize time taken by platoon to equilibrate

Design implications (autonomous vehicles)
Offer design guidelines to (i) stabilize traffic flow, (ii) increase resource utilization, and (iii) offer better ride quality (lack of jerky motion)
Car-following models: A delayed dynamical view of transportation networks
Model representations

### Pictorial

![Diagram showing vehicle positions and distances](image)

### Symbolic

- $x_i$: position of $i^{th}$ vehicle from fixed reference
- $\dot{x}_i$: velocity of $i^{th}$ vehicle
- $\ddot{x}_i$: acceleration of $i^{th}$ vehicle
- $y_i$: distance between $i^{th}$ and $(i-1)^{th}$ vehicle; $y_i = x_{i-1} - x_i$
- $v_i$: velocity of $i^{th}$ vehicle relative to $(i-1)^{th}$ vehicle; $v_i = \dot{x}_{i-1} - \dot{x}_i$
Model representations

**Pictorial**

![Diagram showing vehicle positions and distances]

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**Mathematical**

\[
\dot{x}(t) = f(x(t), x(t - \tau_1), \ldots, x(t - \tau_N)), \quad x \in \mathbb{R}^N, \quad f \in C^k
\]
Existing models

Classical car-following model (CCFM)

\[ \ddot{x}_i(t) = \alpha_i \frac{(\dot{x}_i(t))^m (\dot{x}_{i-1}(t-\tau) - \dot{x}_i(t-\tau))}{(x_{i-1}(t-\tau) - x_i(t-\tau))^l}, \quad i = 1, 2, \ldots, N \]

- \( \alpha_i \): sensitivity coefficient of \( i^{th} \) driver
- \( \tau \): common reaction delay
- \( m \in [-2, 2], l \in \mathbb{R}_+ \): model parameters
Existing models

### Classical car-following model (CCFM)

\[ \ddot{x}_i(t) = \alpha_i \frac{(\dot{x}_i(t))^m (\dot{x}_{i-1}(t-\tau) - \dot{x}_i(t-\tau))}{(x_{i-1}(t-\tau) - x_i(t-\tau))^l}, \quad i = 1, 2, \ldots, N \]

- \( \alpha_i \): sensitivity coefficient of \( i^{th} \) driver
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- \( m \in [-2, 2], l \in \mathbb{R}_+ \): model parameters

### Optimal velocity model (OVM)

\[ \ddot{x}_1(t) = a \left( V(x_N(t - \tau) - x_1(t - \tau)) - \dot{x}_1(t - \tau) \right) \]

\[ \ddot{x}_i(t) = a \left( V(x_{i-1}(t - \tau) - x_i(t - \tau)) - \dot{x}_i(t - \tau) \right), \quad i = 2, 3, \ldots, N \]

- \( a \): common sensitivity coefficient
- \( \tau \): common reaction delay
- \( V(\cdot) \): optimal velocity function
Optimal velocity functions

Properties

- **Monotonic increasing**: $y_1 > y_2 \implies V(y_1) > V(y_2)$
- **Upper bounded**: $\exists V^b$ such that $V(y) \leq V^b \quad \forall y$
- **Continuously differentiable**: $V \in C^1(\mathbb{R}_+)$
# Optimal velocity functions

## Properties

- **Monotonic increasing**: \( y_1 > y_2 \implies V(y_1) > V(y_2) \)
- **Upper bounded**: \( \exists V^b \) such that \( V(y) \leq V^b \ \forall y \)
- **Continuously differentiable**: \( V \in C^1(\mathbb{R}_+) \)

## Examples

- **Bando model**: \( V(y) = V_0 \left( \tanh \left( \frac{y-y_m}{\bar{y}} \right) + \tanh \left( \frac{y_m}{\bar{y}} \right) \right) \)
- **Underwood model**: \( V(y) = V_0 e^{-\frac{2ym}{y}} \)
- **Trigonometric model**: \( V(y) = V_0 \left( \tan^{-1} \left( \frac{y-y_m}{\bar{y}} \right) + \tan^{-1} \left( \frac{y_m}{\bar{y}} \right) \right) \)
- **Hyperbolic model**: \( V(y) = \begin{cases} 0, & y \leq y_0, \\ V_0 \left( \frac{(y-y_0)_n}{(\bar{y})^n+(y-y_0)_n} \right), & y \geq y_0. \end{cases} \)

---

Proposed models

Reduced classical car-following model (RCCFM)

\[
\dot{v}_i(t) = \beta_{i-1}(t - \tau_{i-1})v_{i-1}(t - \tau_{i-1}) - \beta_i(t - \tau_i)v_i(t - \tau_i)
\]

\[
\beta_i(t) = \alpha_i (\dot{x}_0(t) - v_0(t) - \cdots - v_i(t))^m, \quad i = 1, 2, \ldots, N
\]
Proposed models

Reduced classical car-following model (RCCFM)

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\[
\beta_i(t) = \alpha_i (\dot{x}_0(t) - v_0(t) - \cdots - v_i(t))^m, \quad i = 1, 2, \ldots, N
\]

Modified optimal velocity model (MOVM)

\[
\dot{v}_1(t) = \ddot{x}_0(t) + a (\dot{x}_0(t - \tau_1) - V(y_1(t - \tau_1)) - v_1(t - \tau_1))
\]

\[
\dot{v}_k(t) = a (V(y_{k-1}(t - \tau_{k-1})) - V(y_k(t - \tau_k)) - v_k(t - \tau_k))
\]

\[
\dot{y}_i(t) = v_i(t), \quad i = 1, 2, \ldots, N, \quad k = 2, 3, \ldots, N
\]
Agenda for analysis

- Non-linear DDEs; hard to analyze $\Rightarrow$ study local properties
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- Determine equilibrium; linearize models about this equilibrium
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- For arbitrary delays, how do systems lose local stability? 
  Hopf bifurcation \(\Rightarrow\) emergence of oscillations (limit cycles)
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- Study linearized systems for (i) no delay, (ii) small delay and (iii) arbitrary delay; use of characteristic equation

- For arbitrary delays, how do systems lose local stability? Hopf bifurcation $\implies$ emergence of oscillations (limit cycles)

- Use higher order terms, center manifold theory and Poincaré normal forms to determine orbital stability of limit cycles and type of Hopf
Local stability analysis

Equilibrium

RCCFM: \( v_i^* = 0 \ \forall i \)

MOVM: \( v_i^* = 0, \ y_i^* = V^{-1}(\dot{x}_0) \ \forall i \)
Local stability analysis

Equilibrium

\[ \text{RCCFM: } v_i^* = 0 \ \forall i \]
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Linearized model

**RCCFM**

\[ \dot{v}_i(t) = \beta^*_{i-1} v_{i-1}(t - \tau_{i-1}) - \beta^*_i v_i(t - \tau_i), \]
\[ \beta^*_i = \alpha_i (\dot{x}_0)^m, \ i \in \{1, 2, \ldots, N\} \]
### Local stability analysis

**Equilibrium**

- **RCCFM:** \( v_i^* = 0 \) \( \forall i \)
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**Linearized model**

- **RCCFM**
  \[
  \dot{v}_i(t) = \beta_{i-1}^* v_{i-1}(t - \tau_{i-1}) - \beta_i^* v_i(t - \tau_i),
  \]
  \[
  \beta_i^* = \alpha_i(\dot{x}_0)^m, \, i \in \{1, 2, \ldots, N\}
  \]

- **MOVM**
  \[
  \dot{v}_i(t) = du_{i-1}(t - \tau_{i-1}) - du_i(t - \tau_i) - av_i(t - \tau_i),
  \]
  \[
  \dot{u}_i(t) = v_i(t), \, d = aV'(y_i^*), \, i \in \{1, 2, \ldots, N\}
  \]
Absence of delays $\implies$ stability

Dynamics without delays

- **RCCFM**
  \[ \dot{v}_i(t) = \beta_{i-1}^* v_{i-1}(t) - \beta_i^* v_i(t), \quad i \in \{1, 2, \ldots, N\} \]

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  Can be expressed as $\dot{x} = Ax$
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- Can be expressed as $\dot{x} = Ax$

Eigenvalues of dynamics matrix

- **RCCFM:** $\lambda_i = -\beta^*_i \ \forall i$
- **MOVM:** $\lambda = \frac{-a \pm \sqrt{a^2 - 4d}}{2}$

$\implies$ RCCFM and MOVM are always locally stable for zero delays
Why? What?

Motivated by self-driven cars; expected to have very small delays
Why? What?

- Motivated by self-driven cars; expected to have very small delays
- Use Taylor’s series expansion on time variable
Dynamics for small delays

Why? What?
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- Replace $x(t - \tau)$ by $x(t) - \tau \dot{x}(t)$
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Conditions for local stability

RCCFM: $\beta_i^* \tau_i < 1 \ \forall i$

MOVM: $\max(a, \tilde{d}) \tau_i < 1 \ \forall i$
Dynamics for small delays

Why? What?

- Motivated by self-driven cars; expected to have very small delays
- Use Taylor’s series expansion on time variable
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Note

- Above conditions are sufficient for local stability
Why? What?

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Conditions for local stability

RCCFM: $\beta_i^* \tau_i < 1 \ \forall i$

MOVM: $\max(a, \tilde{a}) \tau_i < 1 \ \forall i$

Note

- Above conditions are **sufficient** for local stability
- Dependence on reaction delay & parameters $\Rightarrow$ **co-design** essential
Locally stable region for arbitrary delay

Characteristic equation

RCCFM: $\lambda + \beta_i e^{-\lambda \tau_i} = 0$

MOVM: $\lambda^2 + (a\lambda + d)e^{-\lambda \tau_i} = 0$
Locally stable region for arbitrary delay

### Characteristic equation

<table>
<thead>
<tr>
<th>Model</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>RCCFM</td>
<td>$\lambda + \beta^*_i e^{-\lambda \tau_i} = 0$</td>
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</tr>
</tbody>
</table>

### Necessary and sufficient condition for local stability

- **RCCFM**
  \[
  \tau_i < \frac{\pi}{2\beta^*_i} \forall i
  \]

- **MOVM**
  \[
  \tau_i < \frac{1}{\chi} \tan^{-1} \left( \frac{\chi}{d} \right) \quad \forall i,
  \]
  where \( \chi = \sqrt{\frac{a(a + \sqrt{a^2 + 4d^2})}{2}} \)
Locally stable region for arbitrary delay

**Characteristic equation**

RCCFM: \( \lambda + \beta_i^* e^{-\lambda \tau_i} = 0 \)

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**Note**

- Delays increase \( \Longrightarrow \) loss of local stability
Locally stable region for arbitrary delay

**Characteristic equation**

**RCCFM:** \[ \lambda + \beta_i^* e^{-\lambda \tau_i} = 0 \]

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**Necessary and sufficient condition for local stability**

- **RCCFM**
  \[ \tau_i < \frac{\pi}{2 \beta_i^*} \quad \forall i \]

- **MOVM**
  \[ \tau_i < \frac{1}{\chi} \tan^{-1} \left( \frac{\chi}{d} \right) \quad \forall i, \]

  where \( \chi = \sqrt{a(a + \sqrt{a^2 + 4\tilde{d}^2})} \)

**Note**

- Delays increase \( \implies \) loss of local stability
- Zero delay \( \implies \) condition trivially satisfied
Necessary and sufficient condition for local stability

When $\alpha_i \tau_i = c$, a real constant,

$$(\dot{x}_0)^m < \frac{\pi}{2c}$$
Locally stable region: RCCFM

**Necessary and sufficient condition for local stability**

When $\alpha_i \tau_i = c$, a real constant,

\[ (\dot{x}_0)^m < \frac{\pi}{2c} \]

**Stability boundary**

![Stability boundary diagram](image-url)

$m < 0$

$m > 0$
Locally stable region: MOVM

Car-following models: A delayed dynamical view of transportation networks

**SC:** \( \max(a, \tilde{d}) \tau_i < 1 \)

**N&SC:** \( \tau_i < \frac{1}{\chi} \tan^{-1} \left( \frac{\chi}{d} \right) \)
On the stability boundary: MOVM

\[ \tilde{v}, \tilde{y}(t) \times 10^{-3} \]

\[ \text{Time (in seconds)} \]

\[ \tilde{v}_3(t), \tilde{y}_3(t) \]

Car-following models: A delayed dynamical view of transportation networks
Loss of local stability: Hopf bifurcation

**Hopf bifurcation**

- Variation in parameter $\Rightarrow$ topological equivalence destroyed
Hopf bifurcation

- Variation in parameter $\implies$ topological equivalence destroyed
- Hopf bifurcation: conjugate pair of eigenvalues cross imaginary axis
Loss of local stability: Hopf bifurcation

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- Variation in parameter $\implies$ topological equivalence destroyed
- Hopf bifurcation: conjugate pair of eigenvalues cross imaginary axis

**Choice of bifurcation parameter, $\kappa$**
- Non-linear system $\dot{x} = \kappa f(x) = f_\kappa(x)$
Loss of local stability: Hopf bifurcation

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- Non-linear system $\dot{x} = \kappa f(x) = f_\kappa(x)$
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  - Exogenous and non-dimensional
  - Does not affect equilibrium
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- Non-linear system $\dot{x} = \kappa f(x) = f_\kappa(x)$
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  - Captures complex dependence among model parameters
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  - Exogenous and non-dimensional
  - Does not affect equilibrium
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**Transversality condition of Hopf spectrum**

$$\text{Real} \left( \frac{d\lambda}{d\kappa} \right)_{\kappa = \kappa_{cr}} \neq 0$$
Transversality condition

RCCFM

\[
\text{Real} \left( \frac{d\lambda}{d\kappa} \right)_{\kappa = \kappa_{cr}} = \frac{2\beta_i^* \tau_i^2 \omega_0^2}{(2n+1)(1+\tau_i^2 \omega_0^2)\pi} > 0
\]
Transversality condition

**RCCFM**

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\text{Real } \left( \frac{d\lambda}{d\kappa} \right)_{\kappa=\kappa_{cr}} = \frac{2\beta^*_i \tau^2 \omega^2}{(2n+1)(1+\tau^2 \omega_0^2)} \pi > 0
\]

**MOVVM**

\[
\text{Real } \left( \frac{d\lambda}{d\kappa} \right)_{\kappa=\kappa_{cr}} = \frac{\kappa_{cr} \omega^2 \tau_i (\kappa_{cr}^2 \tilde{d} \cos(\omega_0 \tau_i) + \omega_0^2)}{(\kappa_{cr}^2 \tilde{d} \cos(\omega_0 \tau_i) + \omega_0)^2 + (\kappa_{cr}^2 \tilde{d} \sin(\omega_0 \tau_i))^2} > 0
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\]

Implications

- The eigenvalues move to right in Argand plane
Transversality condition

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- The eigenvalues move to right in Argand plane
- Lost stability cannot be regained
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\]

**Implications**

- The eigenvalues move to right in Argand plane
- Lost stability cannot be regained
- Reaction delay is increased \( \Rightarrow \) system loses local stability
Overview

Using Poincaré normal forms and center manifold theory\(^1\)

- supercritical/subcritical Hopf
- orbital stability of limit cycles

Hopf bifurcation analysis

Overview

Using Poincaré normal forms and center manifold theory\[1\]

- supercritical/subcritical Hopf
- orbital stability of limit cycles

Style of analysis

- Let $q$ be complex eigenvector of Jacobian $Df_\kappa(x^*)$ of $\dot{x} = f_\kappa(x)$
- Reduce flow of $f_\kappa(x)$ to a lower-dimensional manifold (center manifold) which is invariant under flow tangential to $q$-plane
- Rewrite dynamics on center manifold using single complex variable
- Determine sign of first Lyapunov coefficient and Floquet exponent to establish type of Hopf and orbital stability of emergent limit cycles

Bifurcation diagrams

Amplitude (relative velocity)

Bifurcation parameter, $\kappa$

RCCFM, $m > 0$

$\begin{align*}
m &= 1 \\
m &= 1.5 \\
m &= 2
\end{align*}$
Bifurcation diagrams

Amplitude (relative velocity) vs Bifurcation parameter, $\kappa$

- $m = -1$
- $m = -1.5$
- $m = -2$

RCCFM, $m < 0$
Bifurcation diagrams

Bando model

Amplitude (relative velocity)

Bifurcation parameter, κ
Bifurcation diagrams

Amplitude (relative velocity)

\[ y_i^* = 1 \]
\[ y_i^* = 2 \]
\[ y_i^* = 3 \]

Bifurcation parameter, \( \kappa \)

Underwood model
Avoiding jerky motion: non-oscillatory convergence

Why? What?

- Oscillations in state variables $\implies$ jerky vehicular motion
Avoiding jerky motion: non-oscillatory convergence

Why? What?

- Oscillations in state variables $\implies$ jerky vehicular motion
- Appropriate parameter design $\implies$ good road ride quality
Avoiding jerky motion: non-oscillatory convergence

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- Mathematically, eigenvalues should be real and negative
Avoiding jerky motion: non-oscillatory convergence

Why? What?

- Oscillations in state variables $\Rightarrow$ jerky vehicular motion
- Appropriate parameter design $\Rightarrow$ good road ride quality
- Mathematically, eigenvalues should be real and negative

Necessary and sufficient condition for non-oscillatory convergence

RCCFM: $\tau_i < \frac{1}{e\beta_i^*} \quad \forall i$

MOVM: $\tau_i < \frac{1}{md} \ln \left( \frac{-a(m+1)}{m^2d} \right) \quad \forall i$
Avoiding jerky motion: non-oscillatory convergence

Why? What?
- Oscillations in state variables $\implies$ jerky vehicular motion
- Appropriate parameter design $\implies$ good road ride quality
- Mathematically, eigenvalues should be real and negative

Necessary and sufficient condition for non-oscillatory convergence

RCCFM: $\tau_i < \frac{1}{e\beta_i^*} \quad \forall i$

MOVM: $\tau_i < \frac{1}{md} \ln \left( \frac{-a(m+1)}{m^2d} \right) \quad \forall i$

How do these conditions compare with stability conditions?
Region of non-oscillatory convergence: MOVM

$$\tau_{noc} = \frac{1}{md} \ln \left( \frac{-a(m+1)}{m^2 d} \right)$$

$$\tau_{cr} = \frac{1}{\chi} \tan^{-1} \left( \frac{\chi}{d} \right)$$
Why? What?

- Vehicle leaves platoon $\implies$ perturbation
Time for platoon equilibration: rate of convergence

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- How long for platoon to equilibrate?
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- Vehicle leaves platoon $\Rightarrow$ perturbation
- How long for platoon to equilibrate?
- Related to rate of convergence
**Time for platoon equilibration: rate of convergence**

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- Obtain “dimensionless” characteristic equation (Let $z = \lambda \tau$)

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- Substitute $z = \psi - \sigma$

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Mathematical formulation\(^1\)
- Obtain “dimensionless” characteristic equation (Let $z = \lambda \tau$)
- Substitute $z = \psi - \sigma$
- Rate of convergence: Largest $\sigma \geq 0$ such that eigenvalues lie in open left half of Argand plane

\[^1\] F. Brauer, Decay rates for solutions of a class of differential-difference equations, 1979
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- Obtain “dimensionless” characteristic equation (Let $z = \lambda \tau$)
- Substitute $z = \psi - \sigma$
- Rate of convergence: Largest $\sigma \geq 0$ such that eigenvalues lie in open left half of Argand plane
- Analytically intractable\textsuperscript{[2]}

\textsuperscript{[1]} F. Brauer, Decay rates for solutions of a class of differential-difference equations, 1979

\textsuperscript{[2]} S. Chong et al., A simple, scalable, and stable explicit rate allocation algorithms for max-min flow control with minimum rate guarantee, 2001
Rate of convergence: RCCFM

\[ \tau^* = \frac{1}{e\beta^*} = \tau_{noc} \]
Rate of convergence: MOVM

Car-following models: A delayed dynamical view of transportation networks
Rate of convergence: MOVM

![Graph showing reaction delay, τ, and sensitivity coefficient, a, with two lines labeled \( \tau_{cr} \) and \( \tau_{noc} \).]
Rate of convergence defined for pair of vehicles
Rate of convergence defined for \textit{pair} of vehicles

Using notion of \textit{settling time}, define time for pair to equilibrate
Rate of convergence defined for pair of vehicles

Using notion of settling time, define time for pair to equilibrate

Given $\epsilon > 0$, $t_i^e(\epsilon)$ be time for $i^{th}$ pair to enter and subsequently remain within $\epsilon$ distance of equilibrium
Rate of convergence defined for pair of vehicles

Using notion of settling time, define time for pair to equilibrate

Given $\epsilon > 0$, $t_i^e(\epsilon)$ be time for $i^{th}$ pair to enter and subsequently remain within $\epsilon$ distance of equilibrium

Time for platoon to equilibrate

$$T_x^e = \sum_{i=1}^{N} t_i^e, \ x \in \{RCCFM, MOV M\}$$
In a nutshell

Problems addressed

Insight into control laws that yield desirable traffic flow
In a nutshell

**Problems addressed**
Insight into control laws that yield desirable traffic flow

**Contributions**
- Phenomenological insight into “phantom jams”
- Characterize region of non-oscillatory convergence
- Characterize time taken by platoon to equilibrate
In a nutshell

**Problems addressed**

Insight into control laws that yield desirable traffic flow

**Contributions**

- Phenomenological insight into “phantom jams”
- Characterize region of non-oscillatory convergence
- Characterize time taken by platoon to equilibrate

**Design implications (autonomous vehicles)**

Offer design guidelines to (i) stabilize traffic flow, (ii) increase resource utilization, and (iii) offer better ride quality (lack of jerky motion)
Future work

- String stability of RCCFM/MOVM
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- Robustness of RCCFM/MOVM to variations in parameter
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- String stability of RCCFM/MOVM

- Robustness of RCCFM/MOVM to variations in parameter

- Effect of delayed acceleration feedback on RCCFM (preliminary results published)
Publications

Published


Submitted

Published


Submitted

In preparation


- G.K. Kamath, K. Jagannathan and G. Raina, “Local stability and Hopf bifurcation of the reduced classical car-following model,” submission planned to *Nonlinear Dynamics*
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