Coordinate Systems

1 Introduction

Electromagnetics is the study of the effects of electric charges in rest and motion. Some fundamental quantities in electromagnetics are scalars while others are vectors. These scalars and vectors can be functions of position and time and they can be completely described using an appropriate coordinate system. The laws of electromagnetism are invariant with coordinate systems i.e., they can be completely described using an appropriate coordinate system and must hold good irrespective of the coordinate system used. The choice of a specific coordinate system is decided by the geometry of the given problem.

There are 8 orthogonal coordinate systems, namely

1. Cartesian Coordinate System
2. Cylindrical Coordinate System
3. Spherical Coordinate System
4. Parabolic Cylindrical Coordinate System
5. Conical Coordinate System
6. Prolate Spheroidal Coordinate System
7. Oblate Spheroidal Coordinate System
8. Ellipsoidal Coordinate System

Of these 8 orthogonal coordinate systems, the most widely used are the Cartesian, cylindrical and spherical coordinate system. We will be using these three systems in all of our discussions.

2 Coordinate Basis

A point in a three dimensional space can be located as the intersection of three surfaces. For example, the corner of a room is the point of intersection of the three walls representing the three planes. When these three planes are orthogonal to each other, we have an orthogonal coordinate system. The three orthogonal surfaces can be represented as $u_i = \text{constant}$ ($i = 1, 2, 3$). These surfaces need not be planes; they could be curved surfaces. The unit vector in the three coordinate directions are called base vectors. Any point in space can be written as a linear combination of these base vectors. We now describe the base vectors of each coordinate system. Usually right handed coordinate systems are chosen to describe the problems in electromagnetics.
2.1 Cartesian System

In the Cartesian system, the 3 base vectors are $\hat{a}_x$, $\hat{a}_y$, and $\hat{a}_z$. Any point in space can be written in the form,

$$\vec{P} = x_1\hat{a}_x + y_1\hat{a}_y + z_1\hat{a}_z$$

where $(x_1, y_1, z_1)$ are the coordinates of the point $P$ in the Cartesian space which is the intersection of the three planes $x = x_1, y = y_1, z = z_1$. The distance of the point from the origin is given by,

$$|\vec{P}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

The figure below depicts the point $P$ in the Cartesian Coordinate System. As you can see $x_1, y_1$ and $z_1$ can also be understood as the perpendicular distance of point $P$ from the YZ, XZ and XY plane.

The differential length, differential surface and differential volume are given by
\[ \overrightarrow{dl} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z \]
\[ \overrightarrow{dS}_x = dydz\hat{a}_x \]
\[ \overrightarrow{dS}_y = dxdz\hat{a}_y \]
\[ \overrightarrow{dS}_z = dxdy\hat{a}_z \]
\[ dV = dx dy dz \]

Fig. 2 shows the differential surfaces and the differential volume.

### 2.2 Cylindrical System

In the cylindrical system, the three base vectors are \( \hat{a}_r, \hat{a}_\phi \), and \( \hat{a}_z \). Any point in space can be written in the form,

\[ \overrightarrow{P} = r_1\hat{a}_r + \phi_1\hat{a}_\phi + z_1\hat{a}_z \]

where \((r_1, \phi_1, z_1)\) are the coordinates of the point \(P\) in the Cylindrical Space. The point \(P\) is the intersection of a circular cylinder surface \(r = r_1\), a half plane containing \(z\)-axis and making an angle \(\phi = \phi_1\) with the XZ plane and a plane \(z = z_1\) parallel to the XY plane. \(\phi_1\) is measured from the positive x-axis and the base vector \(\hat{a}_\phi\) is tangential to the cylindrical surface.

![Figure 3: Cylindrical coordinate system](image)

The distance of the point from the origin is given by,

\[ |\overrightarrow{P}| = \sqrt{r_1^2 + z_1^2} \]

The values of the 3 coordinates vary as follows,

\[ r \in [0, \infty) \]
\[ \phi \in [0, 2\pi) \]
\[ z \in (-\infty, \infty) \]
From fig. 4 we can see that the differential length, differential surface and differential volume are given by

\[ \vec{dl} = dr \hat{a}_r + r d\phi \hat{a}_\phi + dz \hat{a}_z \]
\[ \vec{dS}_r = rd\phi dz \hat{a}_r \]
\[ \vec{dS}_\phi = dr dz \hat{a}_\phi \]
\[ \vec{dS}_z = r d\phi dr \hat{a}_z \]
\[ dV = r dr d\phi dz \]

Figure 4: Differential elements in cylindrical system

2.3 Spherical System

In the Spherical system, the 3 basis are \( \hat{a}_r, \hat{a}_\theta \) and \( \hat{a}_\phi \). Any point in space can be written in the form,

\[ \vec{P} = r_1 \hat{a}_r + \theta_1 \hat{a}_\theta + \phi_1 \hat{a}_\phi \]

where \( (r, \theta, \phi) \) are the coordinates of the point \( P \) in the Spherical Space. A point \( P(r_1, \theta_1, \phi_1) \) in the spherical coordinates is specified as the intersection of the following three surfaces: a spherical surface centered at origin and has a radius \( r_1 \), a right circular cone with its apex at origin and half angle \( \theta_1 \) and a half plane containing z-axis and making an angle \( \phi_1 \) with the XZ plane.

Figure 5: Spherical coordinate system
The distance of the point from the origin is given by,

$$|\vec{P}| = r_1$$

The values of the 3 coordinates vary as follows,

$$r \in [0, \infty)$$
$$\theta \in [0, \pi]$$
$$\phi \in [0, 2\pi]$$

Figure 6: Differential coordinates in spherical system

From fig.6 we can see that the differential length, differential surface and differential volume are given by

$$\vec{dl} = dr \hat{a}_r + r d\theta \hat{a}_\theta + r \sin \theta d\phi \hat{a}_\phi$$
$$\vec{dS}_r = r^2 \sin \theta d\theta d\phi \hat{a}_r$$
$$\vec{dS}_\theta = r \sin \theta dr d\phi \hat{a}_\theta$$
$$\vec{dS}_\phi = r dr d\theta \hat{a}_\phi$$
$$dV = r^2 \sin \theta dr d\theta d\phi$$

3 Properties of Basis Vectors

3.1 Cartesian System

$$a_x a_x = a_y a_y = a_z a_z = 1$$
$$a_x a_y = a_y a_z = a_z a_x = 0$$
$$a_x \times a_y = a_z, a_y \times a_z = a_x, a_z \times a_x = a_y$$

3.2 Cylindrical System

$$a_r a_r = a_\phi a_\phi = a_z a_z = 1$$
$$a_r a_\phi = a_\phi a_z = a_z a_r = 0$$
$$a_r \times a_\phi = a_z, a_\phi \times a_z = a_r, a_z \times a_r = a_\phi$$
3.3 Spherical System

\[ a_r a_r = a_{\theta} a_{\theta} = a_{\phi} a_{\phi} = 1 \]
\[ a_r a_{\theta} = a_{\theta} a_r = a_{\phi} a_r = 0 \]
\[ a_r \times a_{\theta} = a_{\phi}, \quad a_{\theta} \times a_{\phi} = a_r, \quad a_{\phi} \times a_r = a_{\theta} \]

4 Conversions

To convert the coordinates of a point from one system to the another, use the following table.

<table>
<thead>
<tr>
<th>To</th>
<th>From</th>
<th>Rectangular</th>
<th>Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>-</td>
<td>( x = r \cos \phi )</td>
<td>( x = \rho \cos \theta \sin \phi )</td>
<td>( x = r \cos \theta \sin \phi )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( y = r \sin \phi )</td>
<td>( y = \rho \sin \theta \sin \phi )</td>
<td>( y = r \sin \theta \sin \phi )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( z = z )</td>
<td>( z = \rho \cos \phi )</td>
<td>( z = \rho \cos \phi )</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>( r = \sqrt{x^2 + y^2} )</td>
<td>( \phi = \tan^{-1} \frac{y}{x} )</td>
<td>( r = \rho \sin \theta )</td>
<td>( z = \rho \cos \theta )</td>
</tr>
<tr>
<td></td>
<td>( \phi = \tan^{-1} \frac{y}{x} )</td>
<td>( z = z )</td>
<td>( z = \rho \cos \theta )</td>
<td>( \phi = \phi )</td>
</tr>
<tr>
<td>Spherical</td>
<td>( r = \sqrt{x^2 + y^2 + z^2} )</td>
<td>( \phi = \tan^{-1} \frac{y}{z} )</td>
<td>( \rho = \sqrt{r^2 + z^2} )</td>
<td>( \phi = \phi )</td>
</tr>
<tr>
<td></td>
<td>( \theta = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} )</td>
<td>( \phi = \phi )</td>
<td>( \theta = \tan^{-1} \frac{z}{r} )</td>
<td>( - )</td>
</tr>
</tbody>
</table>

Figure 7: Pictorial representation of how to convert coordinates from one system to another

5 Vector Calculus

The basic concepts of calculus like differentiability, continuity etc. can be defined for vector functions as well. For this it is very important to know the differential displacement, area and volume elements. We saw these in the previous sections.

5.1 Scalar and Vector Functions

In electromagnetics, we come across two types of functions

- Scalar functions - Those functions whose values are scalar \( f = f(\vec{P}) \), where \( P \) is a given point in space. Eg. Temperature field in a given volume, pressure field of air in the atmosphere etc.
Vector functions - Those functions whose values are vectors

\[ \overrightarrow{v} = \overrightarrow{v}(\overrightarrow{P}) = \begin{bmatrix} v_1(\overrightarrow{P}), v_2(\overrightarrow{P}), v_3(\overrightarrow{P}) \end{bmatrix} \]

where \( v_1, v_2, v_3 \) are the coordinates representing the direction of the vector at the given point \( \overrightarrow{P} \). Eg. Electric field due to a point charge, normal fields of a surface, tangential fields to a curve etc.

![Normals fields to a surface and tangential fields to a curve](image)

Figure 8: Normals fields to a surface and tangential fields to a curve

Vector and scalar functions may depend on time in addition to their dependencies on space.

5.2 Integrals

There are 3 types of integrals in vector calculus. They are

- **Line Integral** - It is the integral of the tangential component of a vector field \( \overrightarrow{A} \) along a curve \( L \). It is given by \( \int_L \overrightarrow{A} \cdot d\overrightarrow{l} \). Fig. 7 shows the line integral of a vector field (blue) over a closed path (red).

![Line integral](image)

Figure 9: Line integral

- **Surface Integral** - Given a vector field \( \overrightarrow{A} \) continuous in a smooth surface \( S \), the surface integral is the flux of \( \overrightarrow{A} \) through \( S \). It is given by \( \int_S \overrightarrow{A} \cdot d\overrightarrow{S} \). Fig. 10 shows the surface integral of a vector field (red) over a surface (blue).

![Surface integral](image)
5.3 Gradient of a Scalar Field

Gradient of a scalar field $V$, is a vector that represents both magnitude and direction of maximum space rate of change of $V$. The gradient of $V$, $\nabla V$, will always be perpendicular to a constant $V$ surface. If $\vec{A} = \nabla V$, then $V$ is said to be the scalar potential of $\vec{A}$. Consider a scalar function - say temperature distribution inside a room. The magnitude of this function depends on the position. Consider two surfaces on which the magnitude of $V$ is a constant with values $V_1$ and $V_1 + dV$ respectively, where $dV$ indicates a small change in $V$. 

\[
\nabla = \frac{\partial}{\partial x} \hat{a}_x + \frac{\partial}{\partial y} \hat{a}_y + \frac{\partial}{\partial z} \hat{a}_z
\]
Let point \( P_1 \) be in the surface \( V_1 \) and \( P_2 \) be in \( V_1 + dV \); \( P_2 \) is a corresponding vector along the normal vector \( \overrightarrow{dn} \); \( P_3 \) is a point close to \( P_2 \) along another vector \( \overrightarrow{dl} \neq \overrightarrow{dn} \). For the same change \( dV \) in \( V \), the space rate of change \( \frac{dV}{dl} \) is greatest along \( \overrightarrow{dn} \) because \( \overrightarrow{dn} \) is the shortest distance between the two surfaces. Since the magnitude of \( \frac{dV}{dl} \) depends on the direction of \( \overrightarrow{dl} \), \( \frac{dV}{dl} \) is a directional derivative. The vector that represents the magnitude and direction of the maximum space rate of increase of a scalar is the gradient of that scalar.

\[
\text{grad} V = \hat{a}_n \frac{dV}{dn}
\]

It can be shown that the gradient operator in the three coordinate system are

\[
\nabla A = \frac{\partial A_x}{\partial x} \hat{x} + \frac{\partial A_y}{\partial y} \hat{y} + \frac{\partial A_z}{\partial z} \hat{z} \quad \text{Rectangular Coordinates}
\]

\[
= \frac{\partial A_\rho}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial A_\theta}{\partial \theta} \hat{\theta} + \frac{\partial A_\phi}{\partial \phi} \hat{\phi} \quad \text{Cylindrical Coordinates}
\]

\[
= \frac{\partial A_r}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \hat{\phi} \quad \text{Spherical Coordinates}
\]

### 5.4 Divergence of a Vector Field

The spatial derivatives of a vector field are represented through divergence and curl. It is usually convenient to represent vector field variations in space as field lines or flux lines whose directions indicate the direction of these lines. The magnitude of a vector field at a given point is indicated either using the length of these field lines or the density of the field lines in the vicinity of that point.

![Figure 12: Different types of fields](image)

The strength of the vector field is measured by the number of flux lines passing through a unit surface normal to that vector. The net outward flux is represented by \( \oint_S \overrightarrow{A} \cdot d \overrightarrow{S} \), where the integral is over the entire surface that bounds the volume. Divergence of a vector field \( \overrightarrow{A} \) at a given point \( P \) is the outward flux per unit volume, as the volume shrinks about \( P \). Divergence of a vector field is a scalar.

\[
div \overrightarrow{A} = \nabla \cdot A = \lim_{\Delta V \to 0} \frac{\oint_S \overrightarrow{A} \cdot d \overrightarrow{S}}{\Delta V}
\]
Given below is the divergence operator in the 3 coordinate systems.

\[
\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{Rectangular Coordinates}
\]

\[
= \frac{1}{\rho} \frac{\partial (\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_\phi}{\partial z} \quad \text{Cylindrical Coordinates}
\]

\[
= \frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (A_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \quad \text{Spherical Coordinates}
\]

In Fig. 12.(a) and 12.(b) there is no divergence and in Fig.12(c) there is a positive divergence. A positive divergence would imply the presence of a source in the given volume and negative divergence would imply a sink. This source is also referred to as the ‘flow source’ and \( \text{div } \vec{A} \) is a measure of the strength of flow source.

**Divergence theorem** relates the divergence of a vector field \( \vec{A} \) to the surface integral of \( \vec{A} \) over a surface. It is given by

\[
\int_S \vec{A} \cdot d\vec{S} = \int_V \nabla \cdot \vec{A} \, dV
\]

where \( S \) is the surface and \( V \) in the volume enclosed by the surface \( S \).

![Figure 13: Divergence of a field inside the spherical surface is equal to the surface integral of the field on the surface](image)

This applies to any surface volume \( V \) that is bounded by the surface \( S \). Direction of \( d\vec{S} \) is always that of the outward normal perpendicular to the surface \( d\vec{S} \) and directed away from the volume.

5.4.1 **Curl**

Similar to a flow source, vector fields can also exist as ‘vortex sources’ which causes circulation if a vector field around it.

Net circulation of a vector field around a closed path is defined as, \( \oint_C \vec{A} \cdot d\vec{l} \), the closed line integral over the path. If \( \vec{A} \) is a force acting on an object, circulation would be the work done by the force in moving th object once around the contour.
The strength of the vortex source can be defined through curl of the vector $\vec{A}$. Circulation is a line integral of a dot product, whose value depends on the orientation of the contour $C$, with respect to the vector $\vec{A}$. To define point function, the contour $C$ is shrunk and oriented such that the circulation is maximum.

Curl of $\vec{A}$ is a vector whose magnitude is the maximum circulation of $\vec{A}$ per unit area, as the area tends to zero. Direction of the curl is the normal direction of the area when the area is oriented to make the circulation maximum.

$$\text{curl} \vec{A} = \nabla \times \vec{A} = \lim_{\Delta S \to 0} \left[ \oint_C \vec{A} \cdot d\ell / \Delta S \right]_{\text{max}}$$

**Stokes Theorem** relates the curl of a vector field $\vec{A}$ to the line integral of $\vec{A}$ over a contour $C$. It is given by

$$\oint_C \vec{A} \cdot d\ell = \int_S (\nabla \times \vec{A}) \cdot d\vec{S}$$

The surface integral of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

![Figure 14](image)

Figure 14: Surface integral of the field on the surface is equal to the line integral of the field along the contour enclosing the surface.

Given below is the $\nabla$ operator in the 3 coordinate systems.

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Rectangular Coordinates

$$= \frac{1}{r} \begin{vmatrix} \hat{r} & r \hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$

Cylindrical Coordinates

$$= \begin{vmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r \sin \theta A_\phi \end{vmatrix}$$

Cylindrical Coordinates