EE 613 Estimation Theory - HW 10

October 28, 2008

1. (10.10) In this problem we discuss reproducing PDFs. If $f(\theta)$ is chosen so that when multiplied by $f(\mathbf{x}|\theta)$ we obtain the same form of PDF in θ, then the posterior PDF $f(\theta|\mathbf{x})$ will have the same form as $f(\theta)$. Now assume that the PDF of $x[n]$ conditioned on θ is the exponential PDF

$$f(x[n]|\theta) = \begin{cases} \theta \exp(-\theta x[n]) & x[n] > 0 \\ 0 & x[n] < 0, \end{cases}$$

where the $x[n]$'s are conditionally independent. Next, assume the gamma prior PDF

$$f(\theta) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} \exp(-\lambda\theta) & \theta > 0 \\ 0 & \theta < 0. \end{cases}$$

where $\lambda > 0$, $\alpha > 0$, and find the posterior PDF. Compare it to the prior PDF. Such a PDF, in this case the gamma, is termed a *conjugate prior* PDF.

2. The data model is $x[n] = a\cos 2\pi f_0 n + b \sin 2\pi f_0 n + \omega[n]$ for $n = 0, 1, \ldots, N-1$ where f_0 is a multiple of $1/N$, excepting 0 or 1/2 (for which $\sin 2\pi f_0 n$ is identically zero), and $\omega[n]$ is WGN with variance σ^2. It is desired to estimate $\boldsymbol{\theta} = [a\ b]^T$, where a and b are random variable with prior PDF $\boldsymbol{\theta} \sim \mathcal{N}(\mathbf{0}, \sigma_\theta^2 \mathbf{I})$, and $\boldsymbol{\theta}$ is independent of $\omega[n]$. Find the MMSE estimate of $\boldsymbol{\theta}$.

3. (11.3) For the posterior PDF

$$f(\theta|x) = \begin{cases} \exp[-(\theta - x)] & \theta > x \\ 0 & \theta < x \end{cases}$$

find the MMSE and MAP estimators.

4. (11.4) The data $x[n] = A + \omega[n]$ for $n = 0, 1, \ldots, N-1$ are observed. The unknown parameter A is assumed to hav the prior PDF

$$f(A) = \begin{cases} \lambda \exp(-\lambda A) & A > 0 \\ 0 & A < 0 \end{cases}$$

where $\lambda > 0$, and $\omega[n]$ is WGN with variance σ^2 and is independent of A. Find the MAP estimator of A.

5. (11.16) In fitting a line through experimental data, we assume the model

$$x[n] = A + Bn + \omega[n] \qquad -M \le n \le M$$

where $\omega[n]$ is WGN with variance σ^2. If we have some prior knowledge of the slope B and intercept A such as

$$\begin{bmatrix} A \\ B \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} A_0 \\ B_0 \end{bmatrix}, \begin{bmatrix} \sigma_A^2 & 0 \\ 0 & \sigma_B^2 \end{bmatrix} \right)$$

find the MMSE estimator of A and B as well as the minimum MSE. Assume that A, B are independent of $\omega[n]$.

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