HW7 solutions (draft)

Q1)

(a)

$$
\begin{aligned}
X(s) &= \frac{s+2}{s^2+8s+15} \\
&= \frac{s+2}{(s+3)(s+5)}
\end{aligned}
$$

Separating the terms in $X(s)$ using partial fractions

$$
\begin{aligned}
X(s) &= \frac{1}{2}\left(\frac{3}{s+5} - \frac{1}{s+3}\right) \\
\implies x(t) &= \frac{1}{2}\left(3e^{-5t} - e^{-3t}\right) u(t)
\end{aligned}
$$

(b)

$$
X(s) = \frac{s+1}{(s+2)^2(s+3)}
$$

Using partial fractions on $X(s)$

$$
X(s) = \frac{A}{s+3} + \frac{B}{s+2} + \frac{C}{(s+2)^2}
$$

Solving for A, B and C,

$$
X(s) = \frac{-2}{s+3} + \frac{2}{s+2} + \frac{-1}{(s+2)^2}
$$

Using the differentiation property of Laplace transform,

$$
\begin{aligned}
t \times y(t) &\longleftrightarrow -\frac{\mathrm{d}Y(s)}{\mathrm{d}s} \\
te^{-2t} &\longleftrightarrow \frac{1}{(s+2)^2}
\end{aligned}
$$

$$
\implies x(t) = \left(-2e^{-3t} + 2e^{-2t} - te^{-2t}\right) u(t)
$$

(c)

$$\begin{aligned} X(s) &= \frac{2s^2+s+1}{s(s+2)} \\ &= 2 - \frac{3s-1}{s(s+2)} \end{aligned}$$

Using partial fractions to separate $X(s)$

$$\begin{aligned} X(s) &= 2 - \frac{1}{2}\left(\frac{-1}{s} + \frac{7}{s+2}\right) \\ \implies x(t) &= 2\delta(t) + \frac{u(t)}{2} - \frac{7}{2}e^{-2t}u(t) \end{aligned}$$

(d)

$$X(s) = \frac{2s+1}{(s+2)(s^2+1)^2}$$

Using partial fractions to split $X(s)$

$$X(s) = \frac{A}{s+2} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{(s^2+1)^2}$$

Solving for A, B, C, D and E,

$$X(s) = \frac{-3}{25(s+2)} + \frac{3s-6}{25(s^2+1)} + \frac{3s+4}{5(s^2+1)^2} \tag{1}$$

Using the differentiation property of Laplace transform,

$$\begin{aligned} t \times y(t) &\longleftrightarrow \frac{\mathrm{d}Y(s)}{\mathrm{d}s} \\ t\sin(t) &\longleftrightarrow -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} \qquad (2) \\ \text{Similarly } t\cos(t) &\longleftrightarrow -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+1}\right) = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2} \\ \implies t\cos(t) - \sin(t) &\longleftrightarrow -\frac{2}{(s^2+1)^2} \qquad (3) \end{aligned}$$

Using the results (2) and (3) in (1),

$$\begin{aligned} x(t) &= \left[\frac{-3}{25}e^{-2t} + \frac{3}{25}\cos(t) - \frac{6}{25}\sin(t) + \frac{3}{10}t\sin(t) - \frac{2}{5}t\cos(t) + \frac{2}{5}\sin(t)\right]u(t) \\ &= \left[\frac{-3}{25}e^{-2t} + \frac{3}{25}\cos(t) + \frac{4}{25}\sin(t) + \frac{3}{10}t\sin(t) - \frac{2}{5}t\cos(t)\right]u(t) \end{aligned}$$

(e)

$$
\begin{aligned}
X(s) &= \frac{1}{10^4 s^2 + 10^2 s + 1} \\
&= \frac{1}{10^4(s^2 + 10^{-2}s + 10^{-4})} \\
&= \frac{1}{10^4\left[\left(s+\frac{1}{200}\right)^2 + \left(\frac{\sqrt{3}}{200}\right)^2\right]} \\
&= \frac{200}{10^4 \times \sqrt{3}} \frac{\frac{\sqrt{3}}{200}}{\left[\left(s+\frac{1}{200}\right)^2 + \left(\frac{\sqrt{3}}{200}\right)^2\right]}
\end{aligned}
$$

Using the relation,

$$
\begin{aligned}
e^{-\alpha t}\sin(\omega t)u(t) &\longleftrightarrow \frac{\omega}{(s+\alpha)^2+\omega^2} \\
x(t) &= \frac{e^{\left(\frac{-t}{200}\right)}\sin\left(\frac{\sqrt{3}t}{200}\right)}{50\sqrt{3}}u(t)
\end{aligned}
$$

Q2)

(a)

Using final value theorem, we can get $v_c(0^+)$ as

$$
\begin{aligned}
v_c(0^+) &= \lim_{s\to\infty} sV_c(s) \\
&= \lim_{s\to\infty} s\frac{as+b}{s^2+cs+d} \\
&= \lim_{s\to\infty} \frac{as^2+bs}{s^2+cs+d} \\
&= \lim_{s\to\infty} \frac{a+\frac{b}{s}}{1+\frac{c}{s}+\frac{d}{s^2}} \\
&= a
\end{aligned}
$$

(b)

Since $i(t) = C\frac{\mathrm{d}V_c}{\mathrm{d}t}$, we get $I(s) = CsV_C(s) - Cv_C(0)$.

$$
\begin{aligned}
I(s) &= Cs\frac{as+b}{s^2+cs+d} - Ca \\
&= \frac{sC(b-ac) - Cad}{s^2+cs+d}
\end{aligned}
$$

Using initial value theorem,

$$
\begin{aligned}
i(0^+) &= \lim_{s\to\infty} sI(s) \\
&= \lim_{s\to\infty} \frac{s^2C(b-ac) - sCad}{s^2+cs+d} \\
&= \lim_{s\to\infty} \frac{C(b-ac) - \frac{Cad}{s}}{1+\frac{c}{s}+\frac{d}{s^2}} \\
&= C(b-ac)
\end{aligned}
$$

(c)

Since $v_L(t) = L\frac{\mathrm{d}i}{\mathrm{d}t}$, we get $V_L(s) = L\left(sI(s) - i(0)\right)$.

$$I(s) = L\left(\frac{s^2C(b-ac) - sCad}{s^2+cs+d} - C(b-ac)\right)$$
$$= LC\left(\frac{s(ac^2-bc-ad) - bd + acd}{s^2+cs+d}\right)$$

Using initial value theorem,

$$v_L(0^+) = \lim_{s\to\infty} sV_L(s)$$
$$= \lim_{s\to\infty} LC\left(\frac{s^2(ac^2-bc-ad) + s(-bd+acd)}{s^2+cs+d}\right)$$
$$= \lim_{s\to\infty} LC\left(\frac{(ac^2-bc-ad) + \frac{(-bd+acd)}{s}}{1+\frac{c}{s}+\frac{d}{s^2}}\right)$$
$$= LC(ac^2-bc-ad)$$

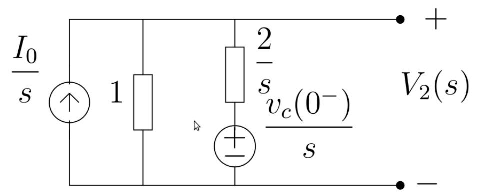
Q3)

At steady state, capacitor is open and the two resistances are in parallel (and $R_{eq} = 1||(1/2) = 1/3$), and

$$v_2 = \frac{I_0}{3}$$

This means, $v_c(0^-) = I_0/3$.

At $t = 0$, the switch K is opened. The new circuit in the transform domain is Now,



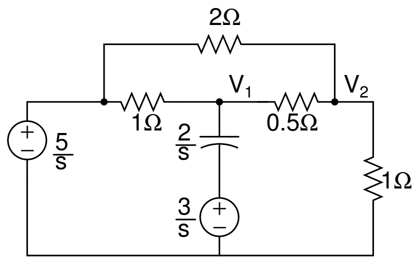
$$\frac{V_2(s) - \frac{V_c(0^-)}{s}}{2/s} + \frac{V_2(s)}{1} = \frac{I_0}{s}$$
$$V_2(s)\left(\frac{s}{2}+1\right) = \frac{I_0}{s} + \frac{V_c(0^-)}{2}$$
$$V_2(s)\left(\frac{s+2}{2}\right) = I_0\left(\frac{6+s}{6s}\right) = \frac{I_0}{3}\left[\frac{3}{s} - \frac{2}{s+2}\right]$$

Therefore, $v_2(t) = I_0\left(1 - \frac{2}{3}e^{-2t}\right)u(t)$ whose values agree with the intuition for the initial and final times.

Q4)

At $t = 0^-$, the capacitor holds a potential same as that of the drop between the 0.5Ω and 1Ω resistors. $v_c(0^-) = \frac{0.5+1}{1+0.5+1} \times 5 = 3V$.

At $t = 0$, when th switch K is closed the circuit in transform domain becomes:



Using KCL at node V_1,

$$\frac{V_1 - \frac{5}{s}}{1} + \frac{V_1 - V_2}{0.5} + \frac{V_1 - \frac{3}{s}}{\frac{2}{s}} = 0 \tag{4}$$

$$\implies V_2 = -\frac{10+3s}{4} + \left(\frac{s+6}{4}\right) V_1 \tag{5}$$

Using KCL for node V_2,

$$\frac{V_2 - \frac{5}{s}}{2} + \frac{V_2 - V_1}{0.5} + \frac{V_2}{1} = 0$$

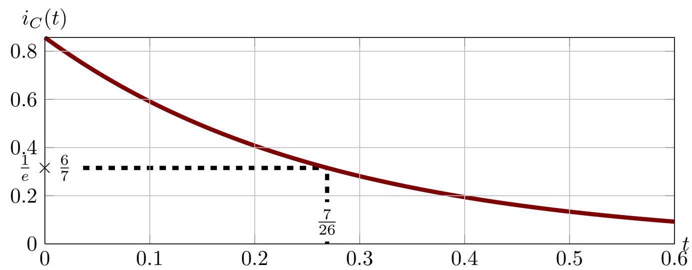
$$\implies V_2 = \frac{5}{7s} + \frac{4}{7} V_1 \tag{6}$$

From equations (5) and (6),

$$V_1(s) = \frac{3\left(s + \frac{30}{7}\right)}{s\left(s + \frac{26}{7}\right)}$$

$$I_c(s) = \frac{V_1(s) - \frac{3}{s}}{\frac{2}{s}} = \frac{\frac{6}{7}}{s + \frac{26}{7}}$$

$$\implies i_c(t) = \frac{6}{7} e^{-\frac{26t}{7}} u(t)$$



Therefore the time constant is $\frac{7}{26}$.

Q5)

At steady state,

$$i(t) = \frac{100}{|Z|} \sin 314t - \theta$$

where $Z = |Z|\angle\theta$ where $|Z| = \sqrt{R^2 + (\omega L - \frac{1}{\omega c})^2}$ and $\theta = \tan^{-1} \frac{(\omega L - \frac{1}{\omega c})}{R}$ and $R = 10^3\Omega$, $L = 1$ H, $C = 1/2\ \mu$F and $\omega = 314$ rad/s. Then, $Z \approx 6137\angle - 80^0$.

When the switch is closed at $t = 0$, $i_L(0^-) = i(t = 0)$ and $v_C(0^-) = v_C(t)|_{t=0} = \frac{100}{|Z|\cdot\omega C}\sin 314t - \theta - 90^0|_{t=0}$. So, $i_L(0^-) = 0.016$ A and $v_C(0^-) = -18.02$ V. The transformed circuit is as shown below:

Using $\omega_0^2 = \frac{1}{LC} \approx 1414$ rad/s,

$$I(s)\left(sL + \frac{1}{sC}\right) = \frac{100\omega}{s^2+\omega^2} + Li_L(0^-) - \frac{v_C(0^-)}{s}$$

$$\Rightarrow I(s) = \frac{100s\omega}{L(s^2+\omega^2)(s^2+\omega_0^2)} + \frac{i_L(0^-)s}{(s^2+\omega_0^2)} - \frac{v_C(0^-)}{L(s^2+\omega_0^2)}$$

$$= \frac{100\omega}{L(\omega_0^2-\omega^2)}\left[\frac{s}{s^2+\omega^2} - \frac{s}{s^2+\omega_0^2}\right] + \frac{i_L(0^-)s}{(s^2+\omega_0^2)} - \frac{v_C(0^-)}{L(s^2+\omega_0^2)}$$

$$\Rightarrow i(t) = A\cos\omega t - B\cos\omega_0 t + i_L(0^-)\cos\omega_0 t - \frac{v_C(0^-)}{L\omega_0}\sin\omega_0 t$$

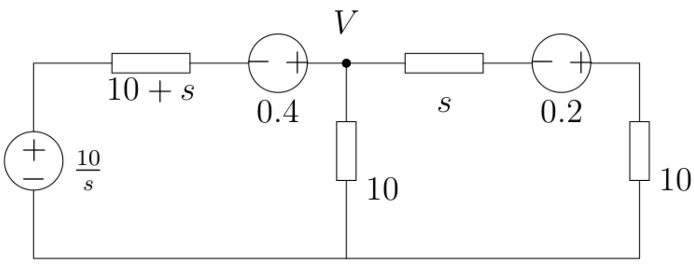
$$= 0.0165\cos\omega t - 0.0165\cos\omega_0 t + 0.016\cos\omega_0 t + 0.012\sin\omega_0 t$$

for $t > 0$. Notice that there are no decaying terms as there is no resistive element in the circuit for all $t > 0$. Check to see if $B = i_L(0^-)$ in which case there will be some cancellation.

Q6)

At steady-state, with the inductances acting as short-circuits, $i_{L_1} = 0.4$ A and $i_{L_2} = s0.2$ A.

At $t = 0$, the 10Ω resistance is shorted. The transform of the resulting circuit is shown below.



Applying KCL at node V,

$$\frac{V - \frac{10}{s} - 0.4}{10+s} + \frac{V}{10} + \frac{V+0.2}{10+s} = 0$$

$$V\left[\frac{1}{10+s} + \frac{1}{10} + \frac{1}{10+s}\right] = \frac{0.2(s+50)}{s(s+10)}$$

$$\Rightarrow V = \frac{2(s+50)}{s(s+30)} = \frac{\frac{10}{3}}{s} - \frac{\frac{4}{3}}{s+30}$$

And, to compute the current,

$$I_R(s) = \frac{V+0.2}{10+s} = \frac{0.2s^2+8s+100}{s(s+10)(s+30)}$$

$$= \frac{\frac{1}{3}}{s} - \frac{\frac{1}{5}}{s+10} + \frac{\frac{1}{15}}{s+30}$$

$$\Rightarrow i_R(t) = \left(\frac{1}{3} - \frac{1}{5}e^{-10t} + \frac{1}{15}e^{-30t}\right)u(t)$$

which checks out OK with initial and final currents.

Q7)

Writing KVL around the 2 loops;

$$v_1(t) = L_1\frac{di_1(t)}{dt} + v_a(t) + M\frac{di_2(t)}{dt} \tag{7}$$
$$v_a(t) = L_2\frac{di_2(t)}{dt} + M\frac{di_1(t)}{dt} \tag{8}$$
$$i_1(t) - i_2(t) = C\frac{dv_a(t)}{dt} \tag{9}$$

Substitute the component values and substitute jω for differentitation since it is sinusoidal steady state.

$$v_1(t) = j\omega i_1(t) + v_a(t) + 0.25j\omega i_2(t) \tag{10}$$
$$v_a(t) = j\omega i_2(t) + 0.25j\omega i_1(t) \tag{11}$$
$$i_1(t) - i_2(t) = j\omega v_a(t) \tag{12}$$

Therefore,

$$v_a(t) = \frac{v_1(t)}{2 - 0.75\omega^2} \tag{13}$$

Since the input signal has a frequency of 1 rad/s ,

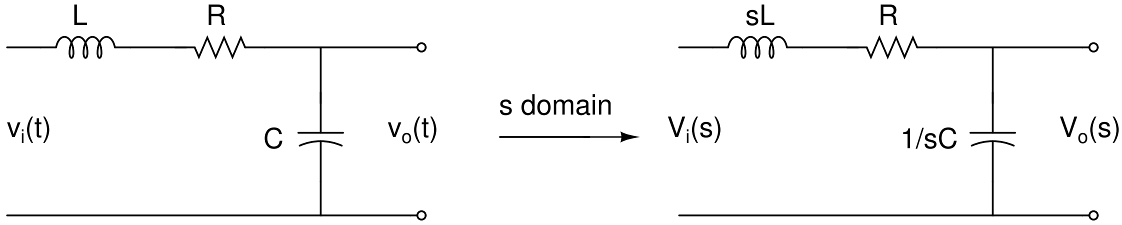
$$v_a(t) = 1.6cos(t) \tag{14}$$

Q8

Given the values $L = 10mH$, $C = 100\mu F$ and $R = 10k\Omega$
Initial conditions are 0.

In the s-domain,

$$L \to sL$$
$$C \to \frac{1}{sC}$$
$$R \to R$$



Applying voltage divider rule to the circuit $V_o(s)$ will be

$$V_o(s) = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} V_i(s)$$
$$\frac{V_o(s)}{V_i(s)} = \frac{1}{s^2LC + sRC + 1}$$

Substituting RLC values, we get

$$\frac{V_o(s)}{V_i(s)} = \frac{10^6}{s^2 + 10^6 s + 10^6}$$