HW5-solutions

Q1)

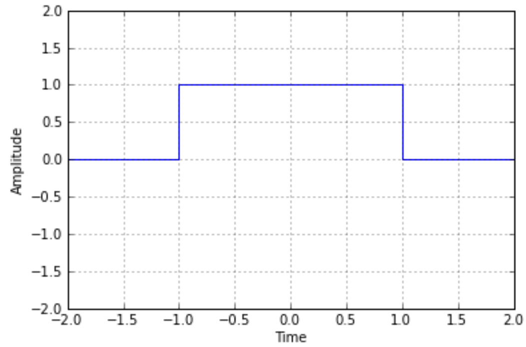


Figure 1: $rect\left(\frac{t}{2}\right)$

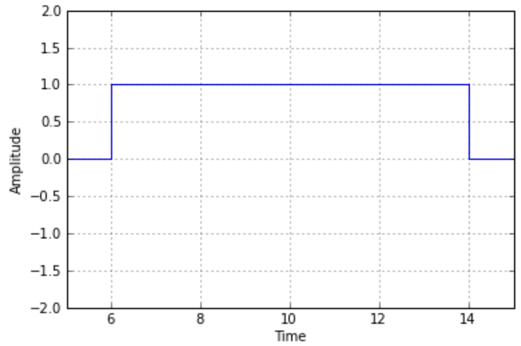


Figure 2: $rect\left(\frac{t-10}{8}\right)$

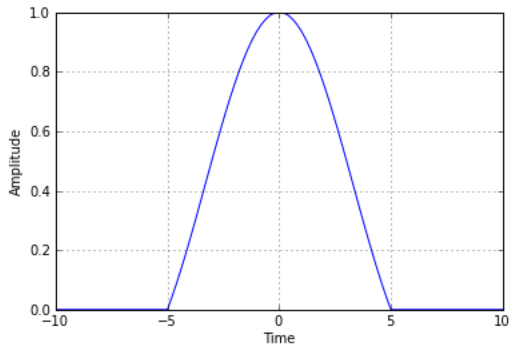


Figure 3: $sinc\left(\frac{t}{5}\right) * rect\left(\frac{t}{10}\right)$

(Q2)

Let $x(t) = \frac{W}{\pi} sinc\left(\frac{Wt}{\pi}\right)$, then,

$$\begin{aligned} X(j\omega) &= 1, \; for|\omega| \le W, \\ &= 0, \; elsewhere \end{aligned} \tag{1}$$

$$X(j\omega) = \int_{-\infty}^{\infty} \frac{W}{\pi} sinc\left(\frac{Wt}{\pi}\right) e^{-j\omega t} dt$$

Let $\tau = \frac{Wt}{\pi}$, then

$$X(j\omega) = \int_{-\infty}^{\infty} sinc(\tau) e^{\frac{-j\omega\pi\tau}{W}} d\tau$$

Now the fourier transform of $sinc(\tau)$ will be a rectangular function of magnitude 1 extending from $-\pi$ to $+\pi$. Let the variable τ be x.

$\int_{-\infty}^{\infty} sinc(x)dx$ is equal to the fourier transform at $\omega = 0$ which is equal to 1.

If $r(t) = s(t)p(t)$, then $R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(\omega-\theta))d\theta$. Using this property,

$$\int_{-\infty}^{\infty} sinc^2(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta = 1$$

Thus, $\int_{-\infty}^{\infty} sinc(x)dx = \int_{-\infty}^{\infty} sinc^2(x)dx = 1$.

Q3)

a

$$x(t) = e^{-\frac{|t|}{2}} \tag{2}$$

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t \\ &= \int_{\infty}^{-\infty} e^{-\frac{|\tau|}{2}} e^{-j\omega\tau}\mathrm{d}\tau \\ &= \int_{-\infty}^{0} e^{\frac{\tau}{2}} e^{-j\omega\tau}\mathrm{d}\tau + \int_{0}^{\infty} e^{-\frac{\tau}{2}} e^{-j\omega\tau}\mathrm{d}\tau \\ &= \frac{1}{\frac{1}{2} - j\omega} + \frac{1}{\frac{1}{2} + j\omega} \\ &= \frac{1}{\frac{1}{4} + \omega^2} \end{aligned} \tag{3}$$

b

The fourier transform property states that multiplication in time domain is convolution in frequency domain.

$$x_1(t) = sin(2\pi t) = \frac{1}{2j}\left(e^{j2\pi t} - e^{-j2\pi t}\right) \tag{4}$$

$$\begin{aligned}
X_1(\omega) &= \frac{1}{2j}\left(\delta(\omega - 2\pi) - \delta(\omega + 2\pi)\right) \\
x_2(t) &= e^{-t}u(t) \\
X_2(\omega) &= \int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}\mathrm{d}t \\
&= \int_0^{-\infty} e^{-\tau}e^{-j\omega\tau}\mathrm{d}\tau \\
&= \frac{1}{1+j\omega} \\
X(\omega) * \delta(\omega - \omega_o) &= X(\omega - \omega_o) \\
y(t) = x_1(t)x_2(t) &\longrightarrow X_1(\omega) * X_2(\omega) \\
Y(\omega) &= \frac{1}{2j}\left(X_2(\omega - 2\pi) - X_2(\omega + 2\pi)\right) \\
&= \frac{2\pi}{1 + (4\pi)^2 - \omega^2 + j2\omega}
\end{aligned} \tag{5}$$

Q4)

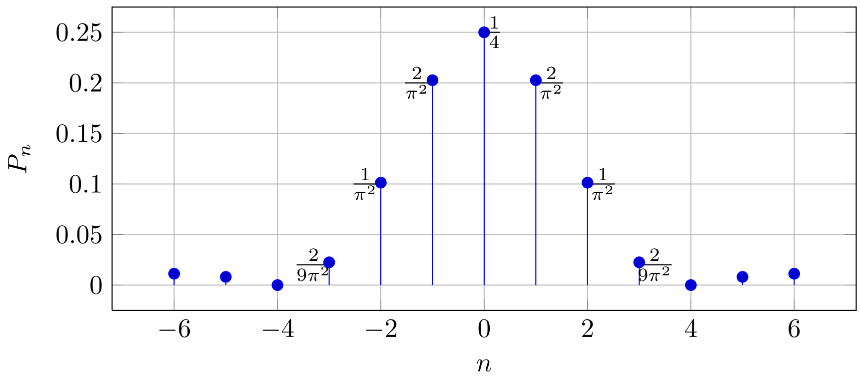
$T_o = 1,\ \omega_o = 2\pi$

(a) Fourier series coefficients P_n of function $p(t)$,

$$\begin{aligned}
P_n &= \frac{1}{T_o}\int_{-1/4}^{3/4} p(t)e^{-jn\omega_o t}\mathrm{d}t,\ \ n \neq 0 \\
&= \int_{-1/4}^{0} (1+4t)e^{-j2\pi nt}\mathrm{d}t + \int_0^{1/4} (1-4t)e^{-j2\pi nt}\mathrm{d}t \\
&= \int_0^{1/4} (1-4t)(e^{-j2\pi nt} + e^{-j2\pi nt})\mathrm{d}t = 2\int_0^{1/4}(1-4t)\cos(2\pi nt)\mathrm{d}t \\
&= 2\left[\frac{\sin(2\pi nt)}{2\pi n} - 4\left(\frac{t\sin(2\pi nt)}{2\pi n} + \frac{\cos(2\pi nt)}{4\pi^2 n^2}\right)\right]_0^{1/4} = \frac{4\sin^2(\pi n/4)}{\pi^2 n^2}
\end{aligned}$$

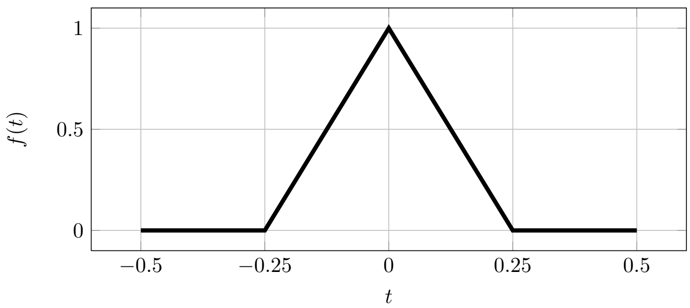
$$P_n = \frac{4\sin^2(\pi n/4)}{\pi^2 n^2}, n \neq 0$$

$$P_0 = \int_{-1/4}^{3/4} p(t)\mathrm{d}t = \frac{1}{4}$$



Fourier transform of $p(t)$:

$f(t)$ is defined as



$p(t)$ is the convolution of $f(t)$ with periodic impulses,

$$\implies p(t) = f(t) * \sum_{k=-\infty}^{\infty} \delta(t - nT_o) \longleftrightarrow P(\omega) = F(\omega) \times \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} \delta(t - nT_o)\right\}$$

Let $h(t) = \sum_{k=-\infty}^{\infty} \delta(t - nT_o)$, Using Fourier series expansion of $h(t)$,

$$h(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_o t}$$

$$c_k = \frac{1}{T_o} \int_{-T_o/2}^{T_0/2} \delta(t) e^{-jk\omega_o t} \mathrm{d}t = 1$$

$$\implies h(t) = \sum_{k=-\infty}^{\infty} e^{j2\pi kt}$$

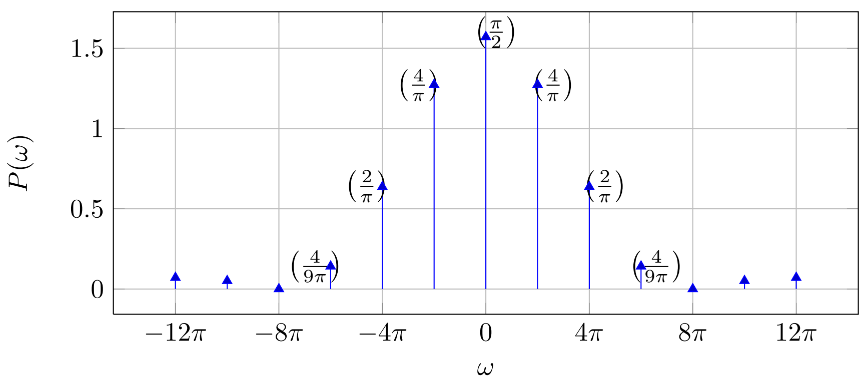
$$\mathcal{F}\left\{h(t)\right\} = \sum_{k=-\infty}^{\infty} \mathcal{F}\left\{e^{j2\pi kt}\right\} = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Using the convolution property of Fourier transform,

$$P(\omega) = F(\omega) \times 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$\begin{aligned}
\text{where } F(\omega) &= \int_{-1/4}^{3/4} f(t)e^{-j\omega t}\mathrm{d}t \\
&= \int_{-1/4}^{0} (1+4t)e^{-j\omega t}\mathrm{d}t + \int_{0}^{1/4} (1-4t)e^{-j\omega t}\mathrm{d}t \\
&= \int_{0}^{1/4} (1-4t)(e^{-j\omega t} + e^{-j\omega t})\mathrm{d}t = 2\int_{0}^{1/4} (1-4t)\cos(\omega t)\mathrm{d}t \\
&= 2\left[\frac{\sin(\omega t)}{\omega} - 4\left(\frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}\right)\right]_0^{1/4} \\
&= 8\left(\frac{1-\cos(\omega/4)}{\omega^2}\right) = \frac{16\sin^2(\omega/8)}{\omega^2}
\end{aligned}$$

$$\begin{aligned}
\implies P(\omega) &= \frac{16\sin^2(\omega/8)}{\omega^2} \times 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \\
&= \sum_{k=-\infty}^{\infty} \frac{32\pi\sin^2(\omega/8)}{\omega^2}\delta(\omega - 2\pi k) = \sum_{k=-\infty}^{\infty} \frac{8\sin^2(\pi k/4)}{\pi k^2}\delta(\omega - 2\pi k)
\end{aligned}$$

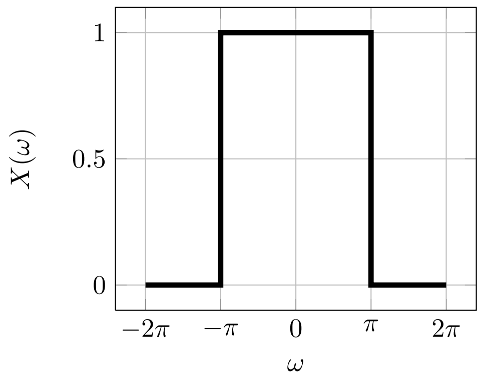


(b)

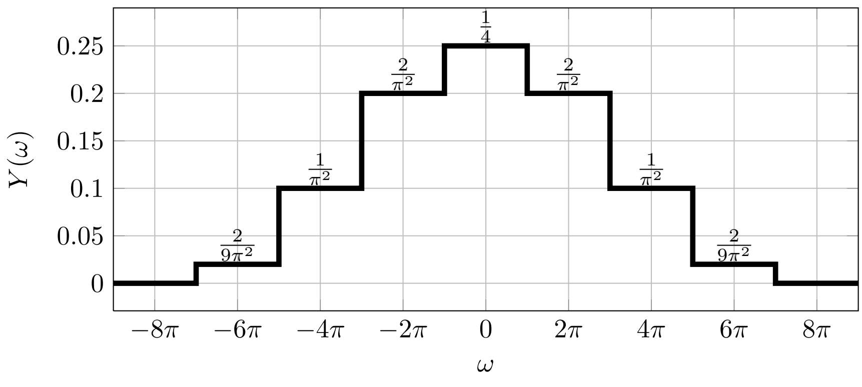
$$y(t) = p(t).x(t) \longleftrightarrow Y(\omega) = \frac{1}{2\pi}P(\omega) * X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\theta)X(\omega-\theta)d\theta$$

(c)

$$x(t) = sinc(t) \longleftrightarrow X(\omega) = rect\left(\frac{\omega}{2\pi}\right)$$



$$Y(\omega) = \frac{1}{2\pi}P(\omega) * X(\omega)$$
$$Y(\omega) = \frac{1}{2\pi}\left[\sum_{k=-\infty}^{\infty}\frac{8\sin^2(\pi k/4)}{\pi k^2}\delta(\omega - 2\pi k)\right] * rect\left(\frac{\omega}{2\pi}\right)$$
$$Y(\omega) = \sum_{k=-\infty}^{\infty}\frac{4\sin^2(\pi k/4)}{\pi^2 k^2}rect\left(\frac{\omega - 2\pi k}{2\pi}\right)$$



Question5

Through integration

$$X(\omega) = \int_{-0.5d}^{0}\frac{2}{d}(t + 0.5d)e^{-j\omega t}dt + \int_{0}^{0.5d}\frac{2}{d}(0.5d - t)e^{-j\omega t}dt \tag{6}$$
$$= \int_{-0.5d}^{0}\frac{2t}{d}e^{-j\omega t}dt + \int_{-0.5d}^{0}e^{-j\omega t}dt + \int_{0}^{0.5d}e^{-j\omega t}dt + \int_{0}^{0.5d}\frac{-2t}{d}e^{-j\omega t}dt \tag{7}$$

Performing integration by parts on the above equation and applying the respective limits gives

$$X(\omega) = \frac{8}{\omega^2 d} sin^2(0.25 d\omega) \tag{8}$$

Time differentiation property Differentiating x(t) gives a waveform whose amplitude is $\frac{2}{d}$ between -0.5d and 0 and the amplitude is $\frac{-2}{d}$ between 0 and 0.5d and 0 otherwise. Let this signal be denoted as s(t).

$$S(\omega) = \int_{-0.5d}^{0} \frac{2}{d} e^{-j\omega t} dt + \int_{0}^{0.5d} \frac{-2}{d} e^{-j\omega t} dt \tag{9}$$

Integrating the above and applying the limits give

$$S(\omega) = \frac{-8}{j\omega d} sin^2(0.25\omega d) \tag{10}$$

From the integration property of FT;

$$X(\omega) = \frac{S(w)}{j\omega} + \pi\delta(\omega)S(0) \tag{11}$$

$$= \frac{8}{\omega^2 d} sin^2(0.25 d\omega) + \pi\delta(\omega) \frac{(-8)(0.25 d\omega)^2}{jd\omega} \tag{12}$$

$$= \frac{8}{\omega^2 d} sin^2(0.25 d\omega) + 0 \tag{13}$$

$$= \frac{8}{\omega^2 d} sin^2(0.25 d\omega) \tag{14}$$

Convolution property of FT Convolving two rectangles having an amplitude of $\sqrt{\frac{2}{d}}$ during the time -0.25d and 0.25d and 0 otherwise will give x(t). If the individual rectangles are denoted by r(t) then;

$$R(\omega) = \int_{-0.25d}^{0.25d} \sqrt{\frac{2}{d}} e^{-j\omega t} dt \tag{15}$$

Evaluating the above integral and applying the limits give

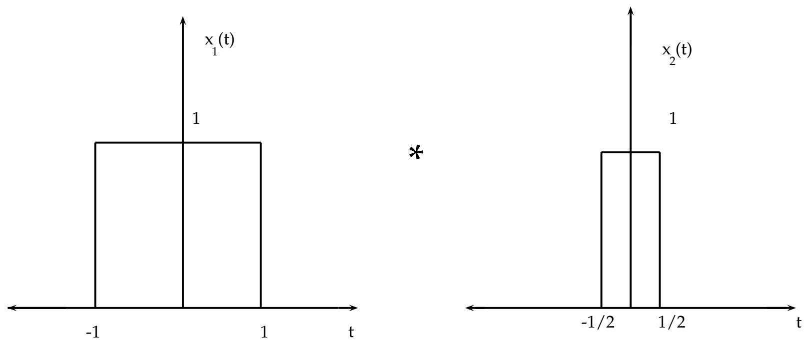
$$R(\omega) = \frac{2\sqrt{2}}{\omega\sqrt{d}} sin(0.25 d\omega) \tag{16}$$

From properties of FT;

$$X(\omega) = (R(\omega))(R(\omega)) \tag{17}$$

$$X(\omega) = \frac{8}{d\omega^2} sin^2(0.25 d\omega) \tag{18}$$

Question6



$x(t) = x_1(t) * x_2(t) \leftrightarrow X(\omega) = X_1(\omega) \times X_2(\omega)$
$X(\omega) = \left(1 \times sinc(\frac{\omega}{2})\right) \times (2 \times sinc(\omega))$
$X(\omega) = 2 \times sinc(\frac{\omega}{2}) \times sinc(\omega)$

Question 7

Energy of the signal $x(t)$ can be found using

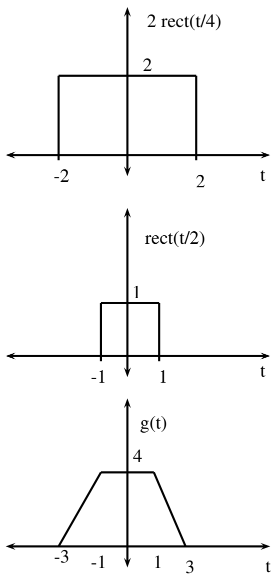
$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt \\
&= \int_{-\infty}^{\infty} e^{-2at} \, u(t) \, dt \\
&= \int_{0}^{\infty} e^{-2at} \, dt \\
&= \left. \frac{e^{-2at}}{-2a} \right|_0^{\infty} \\
E &= \frac{1}{2a} \ \text{J}
\end{aligned}
$$

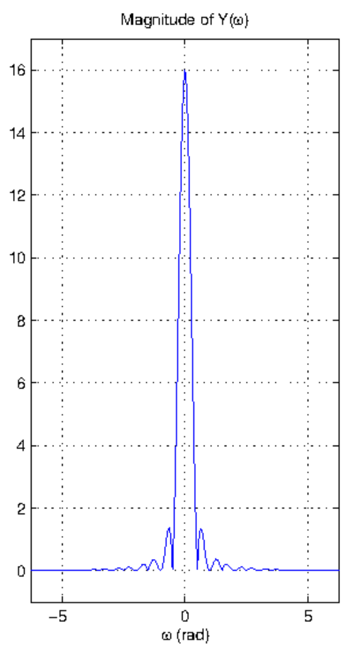
Energy $E_x = 0.95 \times E = \frac{0.95}{2a}$ J
Fourier transform $X(\omega)$ of $e^{at}\ u(t)$ is $\left(\frac{1}{a+j\omega} \right)$
Energy of the signal $x(t)$ using fourier transform

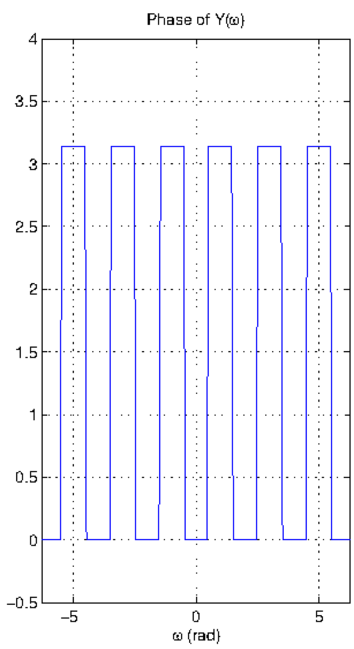
$$
E = \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega
$$

$$
\begin{aligned}
E_x &= \int_{-W}^{W} |X(\omega)|^2 \, d\omega = \int_{-W}^{W} \left| \frac{1}{a + j\omega} \right|^2 \, d\omega \\
&= \int_{-W}^{W} \left(\frac{1}{\sqrt{a^2 + \omega^2}} \right)^2 \, d\omega = \int_{-W}^{W} \frac{1}{a^2 + \omega^2} \, d\omega \\
&= 2 \int_{0}^{W} \frac{1}{a^2 + \omega^2} \, d\omega \\
&= \frac{2}{a} \; \tan^{-1} \, \frac{\omega}{a} \Big|_0^W \\
\frac{0.95}{2a} &= \frac{2}{a} \tan^{-1} \frac{W}{a} \\
\frac{W}{a} &= \tan(0.2375) \\
\frac{W}{a} &= 0.2420 \\
W &= 0.2420 \, a \;\; \text{rad/s}
\end{aligned}
$$

Question8







Using the property:

$$\begin{aligned}
\text{rect}\left(\frac{t}{\tau}\right) &\leftrightarrow \tau\text{sinc}\left(\frac{\omega\tau}{2}\right) && (19)\\
\text{and: } x_1(t) * x_2(t) &\leftrightarrow X_1(\omega)X_2(\omega) && (20)\\
\Rightarrow f_1(t) &\leftrightarrow 2(4)\text{sinc}\,(2\omega)\\
\Rightarrow f_2(t) &\leftrightarrow 2\ \text{sinc}(\omega)\\
\Rightarrow g(t) &\leftrightarrow Y(\omega) = 16\ \text{sinc}(2\omega)\ \text{sinc}(\omega)
\end{aligned}$$

Also, $g(t)$ is given by:

$$\begin{aligned}
g(t) &= t-3, \quad -3 \le t < -1\\
&= 4, \quad -1 \le t < 1\\
&= -2t+6, \quad 1 \le t \le 3
\end{aligned}$$

Question 9

a

$$x_1(t) = rect(t) * rect(t) = x(t+1) + x(-t+1) \tag{21}$$

Fourier transform of $x(-t+1) = X'(\omega)$ is,

$$X'(\omega) = \int_{-\infty}^{\infty} x(-t+1)e^{-j\omega t}\mathrm{d}t \tag{22}$$

Substituting $-t+1$ by τ

$$\begin{aligned}
X'(\omega) &= -\int_{\infty}^{-\infty} x(\tau)e^{-j\omega(2-\tau)}\mathrm{d}\tau\\
&= e^{-j\omega}\int_{-\infty}^{\infty} x(\tau)e^{-j(-\omega)\tau}\mathrm{d}\tau\\
&= e^{-j\omega}X(-\omega) && (23)
\end{aligned}$$

Fourier transform of $x(t+1) = X''(\omega)$ is,

$$X'(\omega) = \int_{-\infty}^{\infty} x(t+1)e^{-j\omega t}\mathrm{d}t \tag{24}$$

Substituting $t+1$ by τ

$$\begin{aligned}
X'(\omega) &= -\int_{\infty}^{-\infty} x(\tau)e^{-j\omega(2-\tau)}\mathrm{d}\tau\\
&= e^{j\omega}\int_{-\infty}^{\infty} x(\tau)e^{-j(-\omega)\tau}\mathrm{d}\tau\\
&= e^{j\omega}X(-\omega) && (25)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
X_1(\omega) &= e^{j\omega}X(\omega) + X(-\omega)e^{-j\omega} \\
&= \frac{e^{j\omega}}{\omega^2}(e^{-j\omega} + j\omega e^{-j\omega} - 1) + \frac{e^{-j\omega}}{\omega^2}(e^{j\omega} - j\omega e^{j\omega} - 1) \\
&= \frac{1}{\omega^2}(2 - e^{-j\omega} - e^{j\omega}) \\
&= \frac{1}{\omega^2}4\sin(\frac{\omega}{2})^2 \qquad (26)
\end{aligned}
$$

b

The fourier transform property states that convolution in time domain is multiplication in frequency domain. Therefore if $R(\omega)$ is the fourier transform of $rect(t)$, then the Fourier transform of $rect(t) * rect(t)$ is $(R(\omega))^2$.

$(R(\omega))^2$ is given by 26. Hence,

$$
R(\omega) = \frac{2\sin(\frac{\omega}{2})}{\omega} \qquad (27)
$$

Using the scaling property of Fourier transform, the transform of $rect(\frac{t}{2}) = X_2(\omega)$ is given by

$$
\begin{aligned}
X_2(\omega) &= 2R(2\omega) \\
&= \frac{2\sin(\omega)}{\omega} \qquad (28)
\end{aligned}
$$

Question 10

(i)

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t \\
X(0) &= \int_{-\infty}^{\infty} x(t)\mathrm{d}t \\
&= \int_{-1}^{0} \mathrm{d}t + \int_{0}^{1}(1-t)\mathrm{d}t + \int_{1}^{2}(t-1)\mathrm{d}t + \int_{2}^{3}\mathrm{d}t \\
X(0) &= 3
\end{aligned}
$$

(ii)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \mathrm{d}\omega$$

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \mathrm{d}\omega$$

$$\implies \int_{-\infty}^{\infty} X(\omega) \mathrm{d}\omega = 2\pi x(0) = 2\pi$$

(iii)

$$x(1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega} \mathrm{d}\omega$$

$$\implies \int_{-\infty}^{\infty} X(\omega) e^{j\omega} \mathrm{d}\omega = 2\pi x(1) = 0$$

(iv) Using Parseval's theorem,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \mathrm{d}\omega = \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t$$

$$\implies \int_{-\infty}^{\infty} |X(\omega)|^2 \mathrm{d}\omega = 2\pi \left[\int_{-1}^{0} 1^2 \mathrm{d}t + \int_{0}^{1} (1-t)^2 \mathrm{d}t + \int_{1}^{2} (t-1)^2 \mathrm{d}t + \int_{2}^{3} 1^2 \mathrm{d}t \right]$$

$$= 2\pi \left[1 + \frac{1}{3} + \frac{1}{3} + 1 \right] = \frac{16\pi}{3}$$

(v)

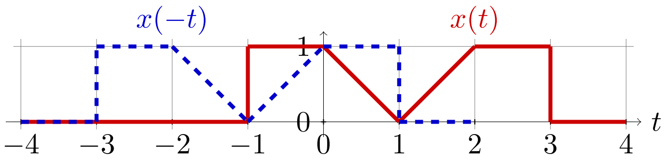
$$\begin{aligned} x(t) &\longleftrightarrow X(\omega) = \mathrm{Re}[X(\omega)] + j\,\mathrm{Im}[X(\omega)] \\ x^*(t) &\longleftrightarrow X^*(-\omega) = \mathrm{Re}[X(-\omega)] - j\,\mathrm{Im}[X(\omega)] \end{aligned}$$

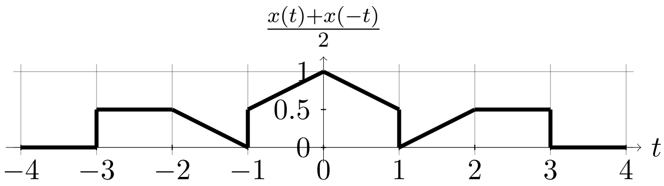
As $x(t)$ is real, $x(t) = x^*(t)$

$$\begin{aligned} \implies \ &\mathrm{Re}[X(\omega)] = \mathrm{Re}[X(-\omega)] \\ \& \ &\mathrm{Im}[X(\omega)] = -\mathrm{Im}[X(-\omega)] \end{aligned}$$

$$\begin{aligned} x(-t) \longleftrightarrow X(-\omega) &= \mathrm{Re}[X(-\omega)] + j\,\mathrm{Im}[X(-\omega)] \\ &= \mathrm{Re}[X(\omega)] - j\,\mathrm{Im}[X(\omega)] \end{aligned}$$

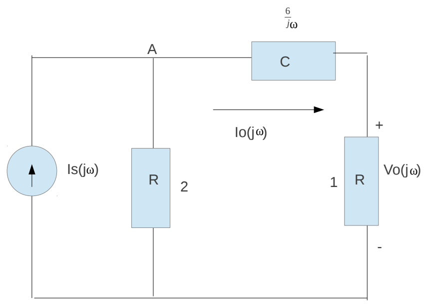
$$\implies \frac{x(t) + x(-t)}{2} \longleftrightarrow \mathrm{Re}[X(\omega)]$$





Question 11

The given ciruit can be drawn equivalently in the frequency domain as below:



Current through the capacitor is

$$I_0(j\omega) = V_0(j\omega)$$

Hence, voltage across the 2Ω resistor is

$$V_0(j\omega)(\frac{6}{j\omega} + 1)$$

Writing Kirchoff's current equation at node A, we get

$$I_0(j\omega) = \frac{V_0(j\omega)(\frac{6}{j\omega} + 1)}{2} + V_0(j\omega)$$

$$\implies I_0(j\omega) = V_0(j\omega)(\frac{3}{j\omega} + \frac{3}{2})$$

Hence, we get the frequency response as

$$Z(j\omega) = \frac{V_0(j\omega)}{I_0(j\omega)} = \frac{1}{\frac{3}{j\omega} + \frac{3}{2}}$$
$$= \frac{2}{3}(\frac{j\omega}{2 + j\omega})$$

For step input, we can write the Fourier transform of output as

$$V_0(j\omega) = Z(j\omega)I_s(j\omega) = \frac{2}{3}(\frac{j\omega}{2 + j\omega})(\frac{1}{j\omega} + \pi\delta(\omega))$$
$$= (\frac{2}{3})(\frac{1}{2 + j\omega})$$

From this, we can get the step response by inverse Fourier transform as

$$V_0(t) = \frac{2}{3}e^{-2t}u(t)$$