HW - 2 solutions (draft)

Q1)

Given $x_1(t) = x_1(t + T_1)$ and $x_2(t) = x_2(t + T_2)$ $x(t) = x_1(t) + x_2(t)$

Let the period of $x(t)$ be T $x(t) = x(t + T)$

$x(t + T) = x_1(t + mT_1) + x_2(t + nT_2)$

$mT_1 = nT_2 = T$ where m, n are least integers.

The period of $x(t)$ is the LCM of T_1 and T_2

The condition for periodic is the ratio $\frac{T_1}{T_2}$ must be a rational number.

Q2)

Formula :

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0) \tag{1}$$

(a) Let $t - \tau = p \Rightarrow \tau = t - p$;

$$\int_{-\infty}^{\infty} \delta(t - p)x(p)dp = \int_{-\infty}^{\infty} \delta(p - t)x(p)dp = \int_{-\infty}^{\infty} x(p)\delta(p - t)dp = x(t)$$

(b) $x(t)$

(c) $t = 0;\ e^0 = 1$

(d)

$$\int_{-\infty}^{\infty} \delta(2t - 3)sin(\pi t)dt = \int_{-\infty}^{\infty} \delta(2(t - \frac{3}{2}))sin(\pi t)dt = \int_{-\infty}^{\infty} \frac{1}{2}\delta(t - \frac{3}{2})sin(\pi t)dt = \frac{1}{2}sin(\frac{3\pi}{2}) = \frac{-1}{2}$$

(e) $t = -3;\ e^3$

(f) $t = 1;\ 1^3 + 4 = 5$

(g) $t = 3;\ x(2 - 3) = x(-1)$

(h) $t = 3; e^{(3-1)} \cos(\frac{(3-5)\pi}{2}) = -e^2$

Q3)

When $a > 0$,

$$\int_{-\infty}^{\infty} f(t)\delta(at)dt = \int_{-\infty}^{\infty} f\left(\frac{t}{a}\right)\delta(t)\frac{1}{a}dt = \int_{-\infty}^{\infty} f(0)\delta(t)\frac{1}{a}dt = \int_{-\infty}^{\infty} f(t)\delta(t)\frac{1}{a}dt$$

When $a < 0$, using b=-a

$$\int_{-\infty}^{\infty} f(t)\delta(-bt)dt = \int_{\infty}^{-\infty} f\left(\frac{-t}{b}\right)\delta(t)\frac{-1}{b}dt = \int_{-\infty}^{\infty} f(0)\delta(t)\frac{1}{b}dt = \int_{-\infty}^{\infty} f(t)\delta(t)\frac{1}{|a|}dt$$

Q4)

(a) $y(t) = \frac{dx(t)}{dt}$

It is linear,time-invariant, causal, unstable and non-ivertible.

Unstable — for any discontinuous signal, derivative at points of discontinuity becomes unbounded.

Non-invertible because for different constants it gives zero.

(b) $y(t) = \int_{-\infty}^{3t} x(\tau)\ d\tau$

It is linear, time-variant, non-causal, unstable and non-invertible.

Time-variant, let $x(t)$ be delayed by t_0, then $y_1(t) = \int_{-\infty}^{3t} x(\tau - to)\ d\tau$

Now delay $y(t)$ by t_0, then $y(t - t_0) = \int_{-\infty}^{3(t-t_0)} x(\tau)\ d\tau$

$y_1(t) \neq y(t - t_0)$

Non-causal, upper limit of the integration is $3t$. For $t > 0$, output depends on the future inputs.

Unstable, lower limit is $-\infty$, for bounded input, output will be undefined.

(c) $y(t) = x(t/2)$

Linear, time-variant, non-causal, stable,invertible.

Non-causal, for $t < 0$, output depends on future input values.

(d)

$$y(t) = \begin{cases} x(t) - x(t - 100) & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Linear, time-variant, causal, stable and Non-invertible.

Time variant because output is always 0 for negative values of time irrespective of the input. Therefore different shifts of the input to the negative time axis will result in non-proportional outputs.

Non-invertible because of many to one mapping. Say, for a DC input the output of the system will be 0 and no inverse system can predict the input.

(e) $\frac{dy}{dt} + 3ty(t) = t^2x(t)$

Linear, time-variant, causal, un-stable and non-invertible.

Unstable as the output keeps increasing with time. [Say, the input signal is such that $\frac{dy}{dt}$ is small. Then, y(t)=tx(t)]

Non-invertible because at t=0, the differential of output is 0. Therefore, the output is independent of the input and therefore an inverse system cannot predicted the original input from the output of this system.

(f) $y(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT)$

Linear, time-variant, non-causal, stable and non-invertible.

Non-invertible, because output will be discrete sequence, from discrete output, continous time input cannot be reconstructed.

(g) $y(t) = x(2t - 4)$, when input is $x(t)$

Linear, time-variant, non-causal, stable and invertible.

Non-causal, Output depends on the future inputs.

Eg: For $t = 5$, $y(5) = x(2 \times 5 - 4) = x(6)$

Q5)

Given $y(t) = \frac{x^2(t)}{dx/d(t)}$

Homogeneity (scalar rule) means that as the strength of input signal is increased (scaled), then the strength of output signal will be also increased (scaled) with same amount.

For the input $Kx(t)$, output should be $Ky(t)$

Let the output of given system is $y_o(t)$ for input $Kx(t)$

$y_o(t) = K\frac{x^2(t)}{dx/d(t)}$

$y_o(t) = Ky(t)$, Therefore the homogeneity rule is satisfied.

Additivity denotes that the output of system can be computed as sum of the responses resulting from each input signal acting alone.

For the input $x_1(t) + x_2(t)$ output should be $y_1(t) + y_2(t)$

Let the output of the system be $y_a(t)$ for the input $x_1(t) + x_2(t)$

$y_a(t) = \frac{(x_1(t)+x_2(t))^2}{d(x_1+x_2)/d(t)}$

$$y_1(t) + y_2(t) = \frac{x_1^2(t)}{dx_1/d(t)} + \frac{x_2^2(t)}{dx_2/d(t)}$$

Here $y_a(t)$ is not equal to $y_1(t) + y_2(t)$.

So the system does not satisy the additivity property.

Q6)

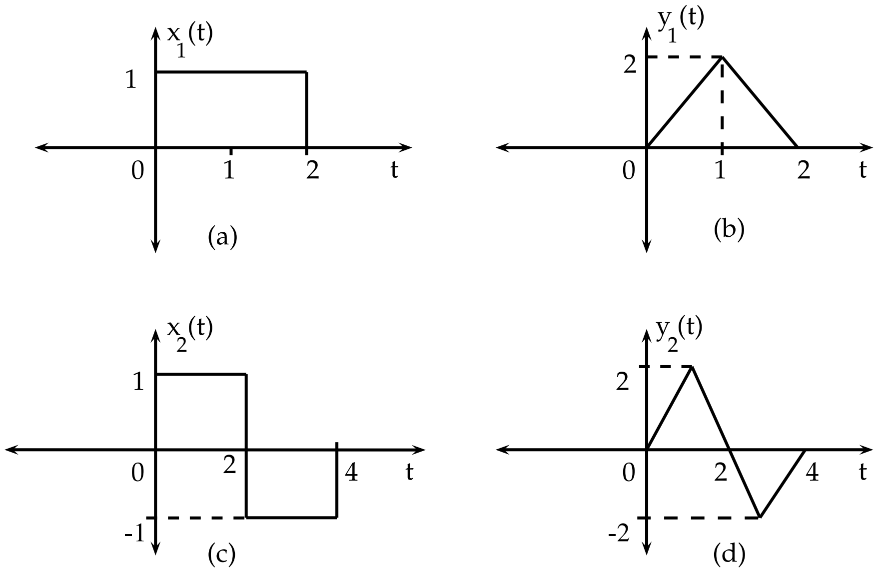


Figure 1: Sketches represent : (a) $x_1(t)$, (b) $y_1(t)$, (c) $x_2(t)$ and (d) $y_2(t)$.

The solution follows from property of LTI systems, i.e. :

$$\text{If: } x_1(t) \rightarrow y_1(t)$$

$$\text{then } x_2(t) \rightarrow y_2(t)$$

$$x_1(t) - x_1(t-2) \rightarrow y_1(t) - y_1(t-2)$$

Q7)

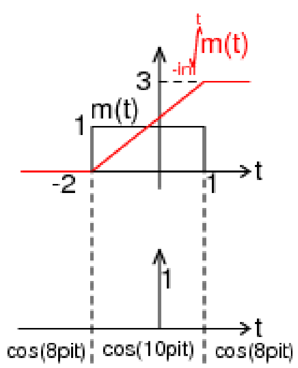
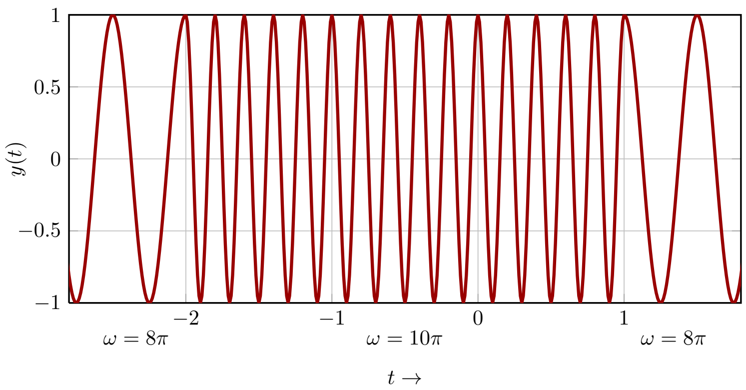


Figure 2: Q7) FM modulation

a

$$y(t) = cos(8\pi t) \text{ for t <-2} \tag{2}$$
$$y(t) = cos(10\pi t + 4\pi) = cos(10\pi t) \text{ for -2 <t <1} \tag{3}$$
$$y(t) = cos(8\pi t + 6\pi) = cos(8\pi t) \text{ for 1 <t} \tag{4}$$
$$y(t)|_{t=-2} = 1 \tag{5}$$
$$y(t)|_{t=1} = 1 \tag{6}$$
$$\tag{7}$$



Using the above equations the time domain FM modulated waveform can be drawn.

b Non-linear: output amplitude does not scale with input amplitude.

Time-variant: If the input is shifted by T,

$$m(t) \to y(t) = Acos\left(\omega_c t + \omega_\Delta \int_{-\infty}^{t} m(\tau)d\tau\right)$$
$$m(t+T) \to y'(t) = Acos\left(\omega_c t + \omega_\Delta \int_{-\infty}^{t+T} m(\tau)d\tau\right)$$

But

$$y(t+T) = Acos\left(\omega_c t + \omega_c T + \omega_\Delta \int_{-\infty}^{t+T} m(\tau)d\tau\right)$$

$\implies y(t) \neq y'(t) \therefore$ the system is time-variant

Has Memory: Integration operation on the message cause the output at the present instant to depend on past inputs of the message signal.

Causal: The output at present instant does not depend on future values of input.

Q8)

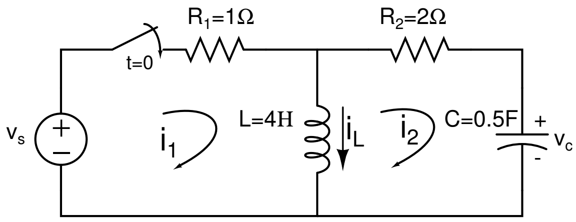


Figure 3: Q8) Input v_s and output i_1

Using KVL in the two loops,

$$L\frac{\mathrm{d}(i_1 - i_2)}{\mathrm{d}t} = v_s - i_1 R_1 \tag{8}$$

$$i_2 R_2 + \frac{1}{c}\int i_2 \mathrm{d}t = v_s - i_1 R_1 \tag{9}$$

Solving for i_2 in terms of i_1 from the equations 8 and 9 yields:

$$i_2 = \frac{D^2 L}{(D^2 L + D R_2 + \frac{1}{C})} i_1 \tag{10}$$

where D is the differential operator $\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)$

Substituting (10) in (8) gives

$$\left(D^2 + D\frac{R_2}{L} + \frac{1}{LC}\right) v_s = \left(D^2(R_1 + R_2) + D\left(\frac{1}{C} + \frac{R_1 R_2}{L}\right) + \frac{R_1}{LC}\right) i_1 \tag{11}$$

Using the values of L, C, R_1 and R_2 in 11,

$$2\frac{\mathrm{d}^2 v_s}{\mathrm{d}t^2} + \frac{\mathrm{d}v_s}{\mathrm{d}t} + v_s = 6\frac{\mathrm{d}^2 i_1}{\mathrm{d}t^2} + 5\frac{\mathrm{d}i_1}{\mathrm{d}t} + i_1 \tag{12}$$