Note Title

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Problem: Given $|H(e^{j\omega})|^2$, how can we get $H(z)$ under the condition that $H(z)$ is assumed to be rational?

Let $H(z)$ be of the form $\dfrac{B(z)}{A(z)}$ where the polynomial coefficients are real-valued. Using the factored form, we saw that

$$H(z)\,H^*(1/z^*) = b_0^2 \, \frac{\prod_{\ell=1}^{M} (1 - c_\ell z^{-1})(1 - c_\ell^* z)}{\prod_{k=1}^{N} (1 - d_k z^{-1})(1 - d_k^* z)}$$

Since the coefficients are real-valued, the roots occur in conjugate pairs.

Hence the factors related to c_ℓ are:

$$(1-c_\ell z^{-1})(1-c_\ell^* z)(1-c_\ell^* z^{-1})(1-c_\ell z)$$

But, $(1-c_\ell z^{-1})(1-c_\ell z) = 1 - c_\ell (z+z^{-1}) + c_\ell^2$

and $(1-c_\ell^* z^{-1})(1-c_\ell^* z) = 1 - c_\ell^* (z+z^{-1}) + c_\ell^{*2}$

$\Rightarrow$ $H(z)H^*(1/z^*)$ is a function of $z+z^{-1}$

Thus,

$$H(z)H^*(1/z^*) = V(w) \quad \text{where} \quad w = \frac{1}{2}(z+z^{-1})$$

Since $h[n] \in \mathbb{R}$, $H(z) = H^*(z^*)$. Hence $V(w) = H(z)H(1/z)$

Evaluating the above at $z = e^{j\omega}$, we get

$$|H(e^{j\omega})|^2 = \mathcal{V}(\cos\omega) = A^2(\omega)$$

<u>Example</u>

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

$$H(z)\,H^*(1/z^*) = \frac{10 - 3(z + z^{-1})}{\frac{5}{4} - \frac{1}{2}(z + z^{-1})}$$

$$|H(e^{j\omega})|^2 = \frac{10 - 6\cos\omega}{\frac{5}{4} - \cos\omega} = A^2(\omega)$$

Conversely, given $A^2(\omega)$, the steps to get $H(z)$ are:

1) Replace $\cos\omega$ by w to get $V(w)$

2) Find the roots w_i of the num. and den. of $V(w)$.

3) Form the equation $\frac{1}{2}(z+z^{-1}) = w_i$ for each w_i. Let the roots be z_i and $1/z_i$ where z_i denotes the root inside the unit circle.

4) Zeros/Poles of the unknown $H(z)$ are the z_i so obtained.

5) The constant K associated with $H(z)$ is obtained using $H^2(1) = V(1)$

Example

$$A^2(\omega) = \frac{10 - 6\cos\omega}{\frac{5}{4} - \cos\omega}$$

$$V(w) = \frac{10 - 6w}{\frac{5}{4} - w}$$

$w_1 = \frac{5}{3}$ (zero) $\Rightarrow \frac{1}{2}(z + z^{-1}) = \frac{5}{3} \Rightarrow z_1 = \frac{1}{3}$ & $\frac{1}{z_1} = 3$

$w_2 = \frac{5}{4}$ (pole) $\Rightarrow \frac{1}{2}(z + z^{-1}) = \frac{5}{4} \Rightarrow z_2 = \frac{1}{2}$ & $\frac{1}{z_2} = 2$

$H(z) = K \frac{z - 1/3}{z - 1/2}$ $\quad H^2(1) = A^2 \left(\frac{2/3}{1/2}\right)^2 = V(1) = 16 \Rightarrow K = 3$

Thus, $H(z) = \dfrac{3z-1}{z-\frac{1}{2}}$

By construction, since z_i is the root that is inside the unit circle, $H(z)$ has all its poles and zeros inside the unit circle.

A filter whose poles and zeros are inside the unit circle is called as a $\boxed{\text{MINIMUM PHASE}}$ filter.

We will see more about min^{m} phase filters later. The process of getting $H(z)$ from $|H(e^{j\omega})|^2$ is called $\boxed{\text{Spectral Factorization}}$.

Alternate Method:

More insight into the spectral factorization problem can be obtained by considering the following alternate approach.

Consider the mapping $Z = \frac{\eta - 1}{\eta + 1}$

(i) For $\eta = e^{j\omega}$, $Z = j \tan \frac{\omega}{2}$ $\Rightarrow$ as ω goes from $-\pi$ to π, Z goes from $-j\infty$ to $+j\infty$. That is the unit circle is mapped to the imaginary axis.

(ii) Let $|z| \leq 1$. It can easily be seen that this region is mapped to the region $Re\{Z\} \leq 0$, i.e., the region inside the unit circle is mapped to the left-half plane.

(iii) Let $|z| > 1$. It can easily be seen that this region is mapped to the region $Re\{Z\} > 0$, i.e., the region outside the unit circle is mapped to the right-half plane.