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Note Title 25-09-2013

Can there be discontinuities in the phase response other than π ?

Recall that

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)} = H_R(e^{j\omega}) + j H_I(e^{j\omega})$$

$$|H(e^{j\omega})|^2 = H_R^2(e^{j\omega}) + H_I^2(e^{j\omega})$$

$$\theta(\omega) = \begin{cases} \tan^{-1} \dfrac{H_I(e^{j\omega})}{H_R(e^{j\omega})} & H(e^{j\omega}) \neq 0 \\ \text{undefined} & H(e^{j\omega}) = 0 \end{cases}$$

By definition, $\theta(\omega) \in (-\pi, \pi]$ i.e., $-\pi < \theta(\omega) \leq \pi$

Note that $e^{j\omega} \in$ RoC of $H(z)$ and $H(z)$ is a rational transfer function

Hence $H_R(e^{j\omega})$ and $H_I(e^{j\omega})$ are continuous functions of ω.

Discontinuities in $\theta(\omega)$ will occur in the following two cases:

(i) At points where $H_I(e^{j\omega_0}) = 0$ and $H_R(e^{j\omega_0}) < 0$, $\theta(\omega_0) = \pi$.

If $H_I(e^{j\omega_0^-}) < 0$ or $H_I(e^{j\omega_0^+}) < 0$ (or both), then

$\theta(\omega_0^-) = -\pi$, $\theta(\omega_0^+) = -\pi$. Hence, the phase jumps by 2π,

i.e., $\pi - (-\pi) = 2\pi$.

(ii) If $H(e^{j\omega_0}) = 0$, then $\theta(\omega_0)$ is undefined and hence phase cannot be continuous at that point.

The no. of points at which the phase can become discontinuous is finite because $H(z)$ is rational.

Jumps of 2π in $\theta(\omega)$ can be removed by adding or subtracting integer multiples of 2π - called $\boxed{\text{PHASE UNWRAPPING}}$

If we define $\theta(\omega)$ suitably at points where $H(e^{j\omega}) = 0$, is it possible to get rid of discontinuities in $\theta(\omega)$?

The answer is NO because we cannot get rid of jumps of π (odd multiples) by phase unwrapping.

Nevertheless, there is a way to make the phase continuous for systems with rational transfer functions.

Crossing a zero on the unit circle introduces a sign change. But $|H(e^{j\omega})|$ is constrained to be non-negative. Hence the phase is forced to jump by π.

If we replace $|H(e^{j\omega})|$ by $A(\omega)$, where $A(\omega) \in \mathbb{R}$, then the change of sign can be absorbed in $A(\omega)$ and the phase can remain continuous.

Hence, we decompose the frequency response as

$$H(e^{j\omega}) = \underbrace{A(\omega)}_{\text{real-valued}} e^{j\overbrace{\phi(\omega)}^{}}$$

$\boxed{\text{continuous}}$ phase function

real-valued, i.e., can take on both +ve and -ve values.

The decomposition $A(\omega) e^{j\phi(\omega)}$ is not unique because $A(\omega)\, e^{j\phi(\omega)}$ is the same as $-A(\omega)\, e^{j(\phi(\omega)+\pi)}$.

The decomposition can be made unique if we enforce the following constraint:

$$0 \leq \phi(0) < \pi$$

Example

Recall the example where $h[n] = 1 \quad 0 \leq n \leq 2N$

$$H(e^{j\omega}) = e^{-j\omega N} \frac{\sin (2N+1)\,\omega/2}{\sin \omega/2}$$

The usual magnitude-phase decomposition resulted in a $\theta(\omega)$ that had jumps of π at the zero crossings. If we replace $|H(e^{j\omega})|$ by $A(\omega)$ where

$$A(\omega) = \frac{\sin (2N+1)\,\omega/2}{\sin \omega/2}$$

then $\phi(\omega) = -N\omega$, which is now continuous. Note that $A(\omega)$ given above now takes on both +ve and -ve values.

Phase response of a single complex pole is the negative of the phase response of a single complex zero.

Overall response is the result of the responses due to the individual poles and zeros.

Take a look at Example 5.10 in Oppenheim and Schafer's, "Discrete-Time Signal Processing" (2nd edition)

See also MATLAB's UNWRAP and ANGLE commands.

When we were discussing causal signals, we saw that $H_R(e^{j\omega})$ and $H_I(e^{j\omega})$ are not independent but related. Does it mean $|H(e^{j\omega})|$ and $\theta(\omega)$ are also related ?

Consider

$$H_1(z) = 1 - a z^{-1} = 1 - r e^{j\theta} z^{-1}$$

$$|H_1(e^{j\omega})|^2 = 1 + r^2 - 2r\cos(\omega - \theta)$$

$$\theta_1(\omega) = \tan^{-1} \frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}$$

Now let $H_2(z) = -a^* + z^{-1} = -re^{-j\theta} + z^{-1}$

$$H_2(e^{j\omega}) = -re^{-j\theta} + e^{-j\omega}$$

$$H_2(e^{j\omega}) \cdot H_2^*(e^{j\omega}) = (e^{-j\omega} - re^{-j\theta})(e^{j\omega} - re^{j\theta})$$

$$= 1 + r^2 - 2r\cos(\omega - \theta)$$

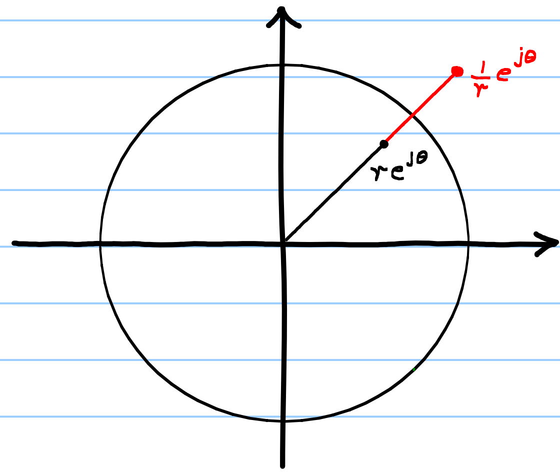
$$= |H_1(e^{j\omega})|^2$$ same magnitude response!

Phase response is different:

$$\theta_2(\omega) = \tan^{-1} \frac{r\sin\theta - \sin\omega}{\cos\omega - r\cos\theta}$$

Zero of $H_1(z)$ is at $re^{j\theta}$

zero of $H_2(z)$ is at $\frac{1}{r}e^{j\theta}$ i.e., the old zero is reflected about the unit circle.



Note: $H_2(z) = -a^* + z^{-1}$

$$= -a^*\left[1 - \frac{1}{a^*}z^{-1}\right]$$

$$= \underbrace{-re^{-j\theta}}_{\text{scale factor needed to make the magnitude identical}}\left[1 - \frac{1}{r}e^{j\theta}z^{-1}\right]$$