Note Title 18-09-2013

EE5330 Sep. 18, 2013

Assigning poles close to the unit circle to get a sharp filter makes the response sensitive to pole location.

Consider $f(x_0+\Delta x) \simeq f(x_0) + \Delta x \cdot f'(x_0)$

$$\Rightarrow \quad f(x_0+\Delta x) - f(x_0) \simeq \Delta x \cdot f'(x_0)$$

$$\Rightarrow \quad \Delta f \simeq \underbrace{\frac{\Delta x}{x_0}}_{\text{relative change in } x_0} x_0 f'(x_0)$$

Δf can become large if $f'(x_0)$ is large. Hence, if $f(\cdot)$ represents the frequency response of a system, $f'(\cdot)$ will be large in the transition region of sharp filters. Such responses are sensitive to small changes in pole locations.

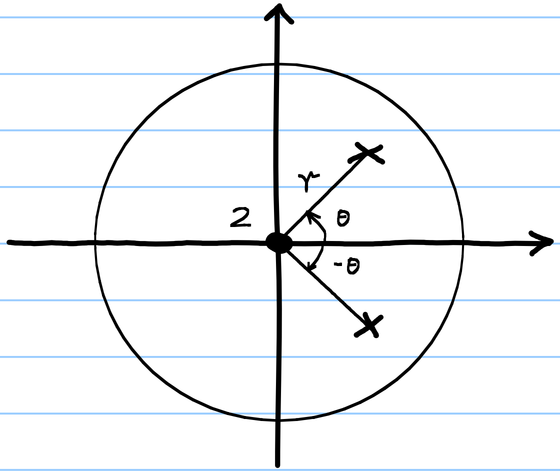
2^{nd} Order Filter:

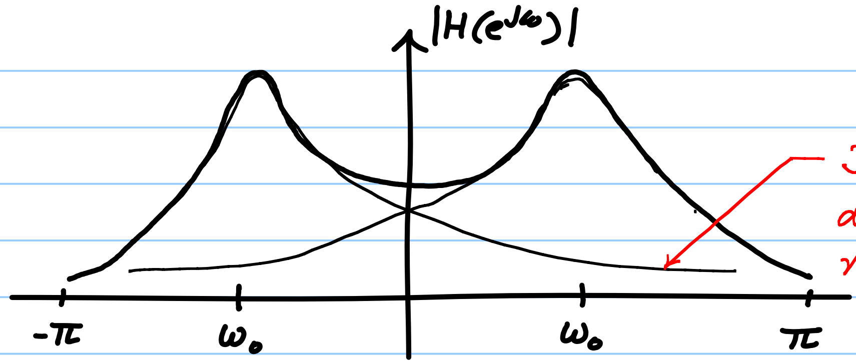
$$h[n] = r^n \frac{\sin[\theta(n+1)]}{\sin\theta} u[n]$$

This filter is also called as a RESONATOR

$$H(z) = \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})} = \frac{1}{1-2r\cos\theta\, z^{-1}+r^2z^{-2}}$$

$$|H(e^{j\omega})|^2 = \frac{1}{[1+r^2-2r\cos(\omega-\theta)][1+r^2-2r\cos(\omega+\theta)]}$$





Tail of the response due to the pole at $re^{-j\theta}$ will interfere and cause a SHIFT in the peak location! In general, the peak will NOT be at $\omega = \pm\theta$.

The expression for the peak location is obtained by solving

$$\omega_0 = \arg\max_{\omega} \frac{1}{[1+r^2-2r\cos(\omega-\theta)][1+r^2-2r\cos(\omega+\theta)]}$$

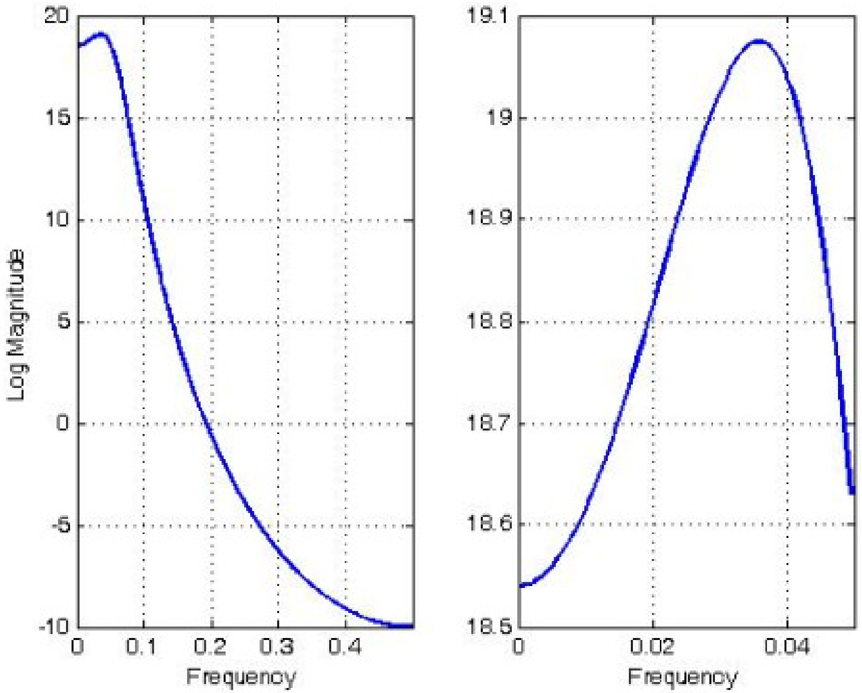
<u>Exercise</u>: Show that $\omega_0 = \cos^{-1}\left[\frac{1+r^2}{2r}\cos\theta\right]$

Interference is reduced if the poles move farther apart. The farthest they can be is when $\theta = \frac{\pi}{2}$. For this value of θ, $\omega_0 = \frac{\pi}{2}$, i.e., there is no shift in peak location for any 'r'!

Interference also reduces as $r \to 1$.

Note that for a distinct peak to be seen at $\omega = \omega_0 \neq 0$, we require $-1 \leq \frac{1+r^2}{2r} \cos\theta \leq 1$

Example of Peak Shifting:



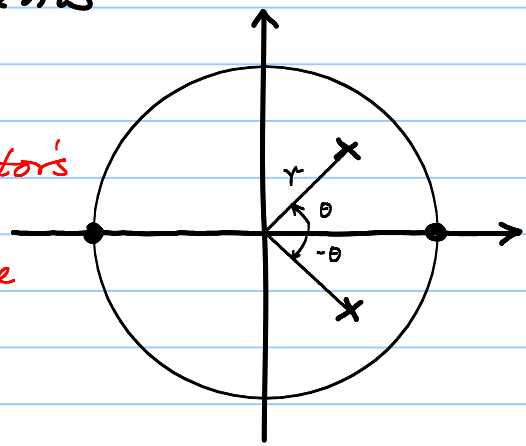
The plots shown on the right correspond to $r = 0.8$ and $\theta = \frac{\pi}{10}$. If there were no interference, the peak would have been at $\omega_0 = \frac{\theta}{2\pi} = 0.05$. However, due to interference, the actual peak occurs at $\frac{\omega_0}{2\pi} = 0.0358$, as can be seen from the plots.

Improved Resonator:

The resonator is a crude Bandpass Filter. A canonic BPF must completely reject frequency components at $\omega = 0$ and $\omega = \pi$. The given resonator can be modified to reject these two frequency components by adding zeros at $z = \pm 1$

$$H(z) = \frac{(1+z^{-1})(1-z^{-1})}{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}}$$

Improved resonator's pole-zero plot.
Note the zeros are now at $z = \pm 1$



The improved resonator also suffers from peak shifting due to tail interference.

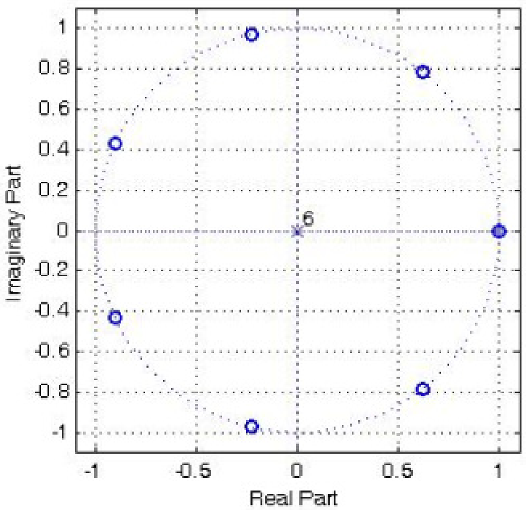
<u>Exercise</u> Derive the expression for the peak location. Comment on the result.

<u>Moving Average Filter</u>:

$$h[n] = \frac{1}{N} \quad 0 \le n \le N-1 \longleftrightarrow H(z) = \frac{1}{N} \frac{1 - z^{-N}}{1 - z^{-1}}$$

$$H(e^{j\omega}) = \frac{e^{-j\omega N/2}}{N} \frac{\sin N\omega/2}{\sin \omega/2}$$

Magnitude Frequency Response shown in both log and linear scales



Pole-Zero Plot of Moving Average Filter