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## Frequency Response of Systems with Rational Transfer Function:

Frequency selective filtering is very important in many practical applications. We can obtain the frequency response by

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

provided the unit circle is part of the RoC, i.e., $e^{j\omega} \in$ RoC

If $e^{j\omega} \in$ RoC, the system is also BIBO stable.

In practice, we will concern ourselves with causal and stable systems. In particular, we will restrict ourselves to the class of

LTI systems characterized by LCCDE.

Some important frequency responses are: LPF, HPF, BPF, BSF, differentiator, and Hilbert transformer.

If the system is to be causal, then ideal, brickwall filters cannot be realized, since they violate the Paley-Wiener theorem.

We will $\boxed{\text{approximate}}$ the ideal responses using rational transfer functions, i.e., by systems that are realizable.

Consider $\quad y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{\ell=0}^{M} b_\ell\, x[n-\ell]$

Taking z-transforms and simplifying,

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{\ell=0}^{M} b_\ell z^{-\ell}}{1 + \sum_{k=0}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

In product form,

$$H(z) = b_0 \frac{\prod_{\ell=1}^{M} (1 - z_\ell z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})} = b_0\, z^{\overbrace{N-M}^{|N-M| \text{ order trivial pole or zero}}} \frac{\prod_{\ell=1}^{M} (z - z_\ell)}{\prod_{k=1}^{N} (z - p_k)}$$

Since the system is stable, $e^{j\omega} \in$ RoC. Hence,

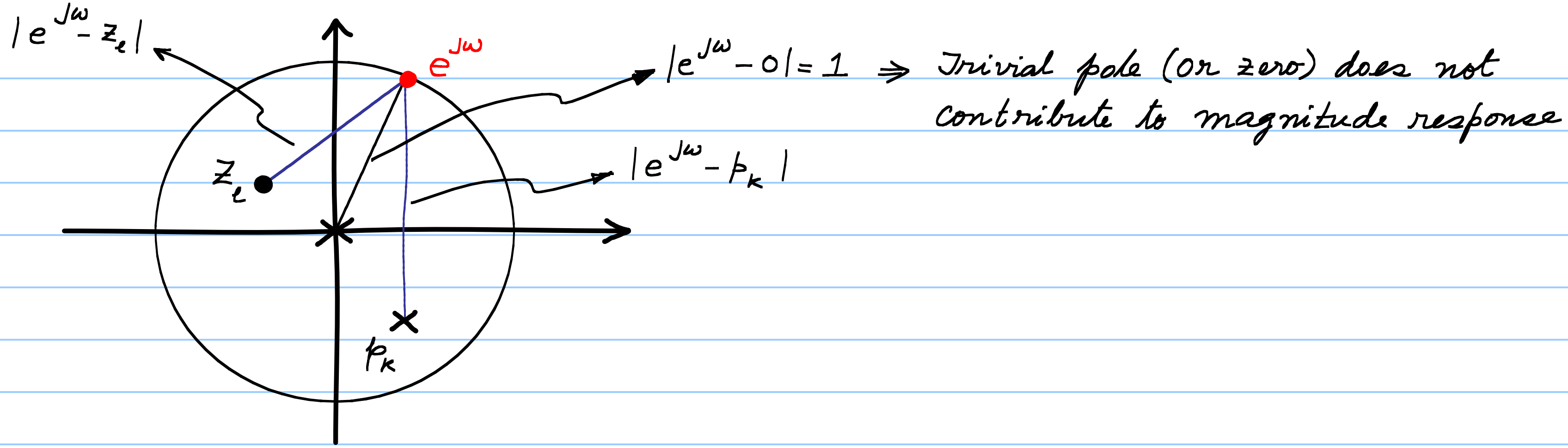
$$H(e^{j\omega}) = b_0 e^{j\omega(N-M)} \frac{\prod_{\ell=1}^{M} (e^{j\omega} - z_\ell)}{\prod_{k=1}^{N} (e^{j\omega} - p_k)} = \underbrace{|H(e^{j\omega})|}_{\text{magnitude response}} e^{j \overbrace{\angle H(e^{j\omega})}^{\text{phase response}}}$$

$$|H(e^{j\omega})| = |b_0| \, \cancelto{1}{\left|e^{j\omega(N-M)}\right|} \, \frac{\prod_{\ell=1}^{M} |e^{j\omega} - z_\ell|}{\prod_{k=1}^{N} |e^{j\omega} - p_k|}$$

Because a pole or zero at $z=0$ does not contribute to the magnitude frequency response, they are called TRIVIAL pole/zero

Trivial poles and zeros contribute to the phase response.

Consider $|e^{j\omega} - z_\ell|$. Geometrically, this denotes the distance from $e^{j\omega}$ (point on the unit circle) to $z_\ell$ (zero at $z = z_\ell$). Thus, the numerator term is the product of all the distances from $e^{j\omega}$ to all the zeros. Similarly, the denominator is the product of all the distances from $e^{j\omega}$ to all the poles. Finally, $|H(e^{j\omega})|$ is the ratio of these two product of distances, multiplied by the gain term $|b_0|$. $|H(e^{j\omega})|$ changes as '$\omega$' changes.




Because $|H(e^{j\omega})|$ spans a large range, we plot the magnitude on a log scale. In particular, we plot $20 \log_{10} |H(e^{j\omega})|$ ( or, equivalently, $10 \log_{10} |H(e^{j\omega})|^2$). The gain term $|b_0|$ merely shifts the curve up or down in the log scale.

The same geometric interpretation holds good in the s-plane also, when interpreting the magnitude of $H(s)\big|_{s=j\Omega}$. For rational $H(s)$,

$$|H(j\Omega)| = |b_0| \frac{\prod_{\ell=1}^{M} |j\Omega - z_\ell|}{\prod_{k=1}^{N} |j\Omega - p_k|}$$

$|H(j\Omega)|$ is the ratio of the product of all the distances from $j\Omega$ to all the zeros to product of all the distances from $j\Omega$ to all the poles, multiplied by $|b_0|$.

The above geometric interpretation reveals that there is no point in the s-plane that is at a constant distance as we move along the $j\Omega$ axis. Hence there is no concept of trivial pole in the s-plane (unlike in the z-plane, where the origin is at a constant distance as we move along the unit circle).