Note Title

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The integrals in the DHT relationships are all Principal Value integrals, i.e.,

$$H_I(e^{j\omega}) = \frac{-1}{2\pi} \, P.V. \int_{-\pi}^{\pi} H_R(e^{j\theta}) \cot\left(\frac{\omega-\theta}{2}\right) d\theta$$

$$= \frac{-1}{2\pi} \lim_{\epsilon \to 0} \left[ \int_{-\pi}^{\omega-\epsilon} (\cdot)\, d\theta + \int_{\omega+\epsilon}^{\pi} (\cdot)\, d\theta \right]$$

## Stability

Let $H(z) = \dfrac{B(z)}{A(z)} = \dfrac{B(z)}{\prod_{k=1}^{N}(1-p_k z^{-1})}$

We know that, if the system is causal, then for stability we require

$|p_k| < 1 \quad \forall k.$

Tests have been devised to check if $|p_k| < 1 \ \forall k$ without explicit roots computation.

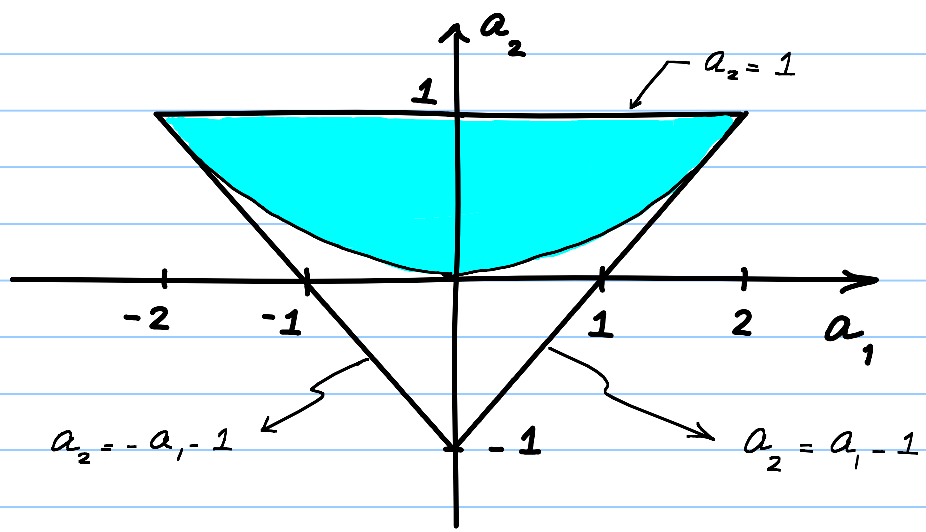
For $2^{nd}$ order systems, we will show that the conditions to be satisfied are :

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2}$$

1) $|a_2| < 1 \Rightarrow -1 < a_2 < 1$

2) $|a_1| < 1 + a_2 \Rightarrow a_1 < 1 + a_2$

$-a_1 < 1 + a_2$




These conditions are satisfied in the triangular region shown on the right, the so-called stability triangle.

In the shaded region, the roots occur in complex conjugate pairs.

We will consider the case of complex conjugates roots first.

$$A(z) = (1 - re^{j\omega_0} z^{-1})(1 - re^{-j\omega_0} z^{-1})$$

$$= 1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}$$

$$= 1 + a_1 z^{-1} + a_2 z^{-2}$$

Stability demands that

$$|a_2| = r^2 < 1$$

$$|a_1| = |2r\cos\omega_0| < 1 + r^2$$

For stability, roots must lie inside the unit circle (assuming causality)

Hence $r < 1 \Rightarrow |a_2| = r^2 < 1$, i.e., the first condition is satisfied.

$$0 < r < 1 \iff (1-r)^2 > 0 \Rightarrow 2r < 1 + r^2$$

$$0 < r < 1 \iff (1+r)^2 > 0 \Rightarrow -2r < 1 + r^2$$

Hence $|2r| < 1 + r^2 \Rightarrow |2r \cos \omega_0| < 1 + r^2$

Thus we have shown that the stability conditions are satisfied iff the complex conjugate roots are inside the unit circle.

Now consider $A(z) = (1 - r_1 z^{-1})(1 - r_2 z^{-1})$ where $-1 < r_i < 1$

$$-1 < r_i < 1 \implies 0 < 1 + r_i < 2$$

$$1 > -r_i > -1 \implies 0 < 1 - r_i < 2$$

Hence

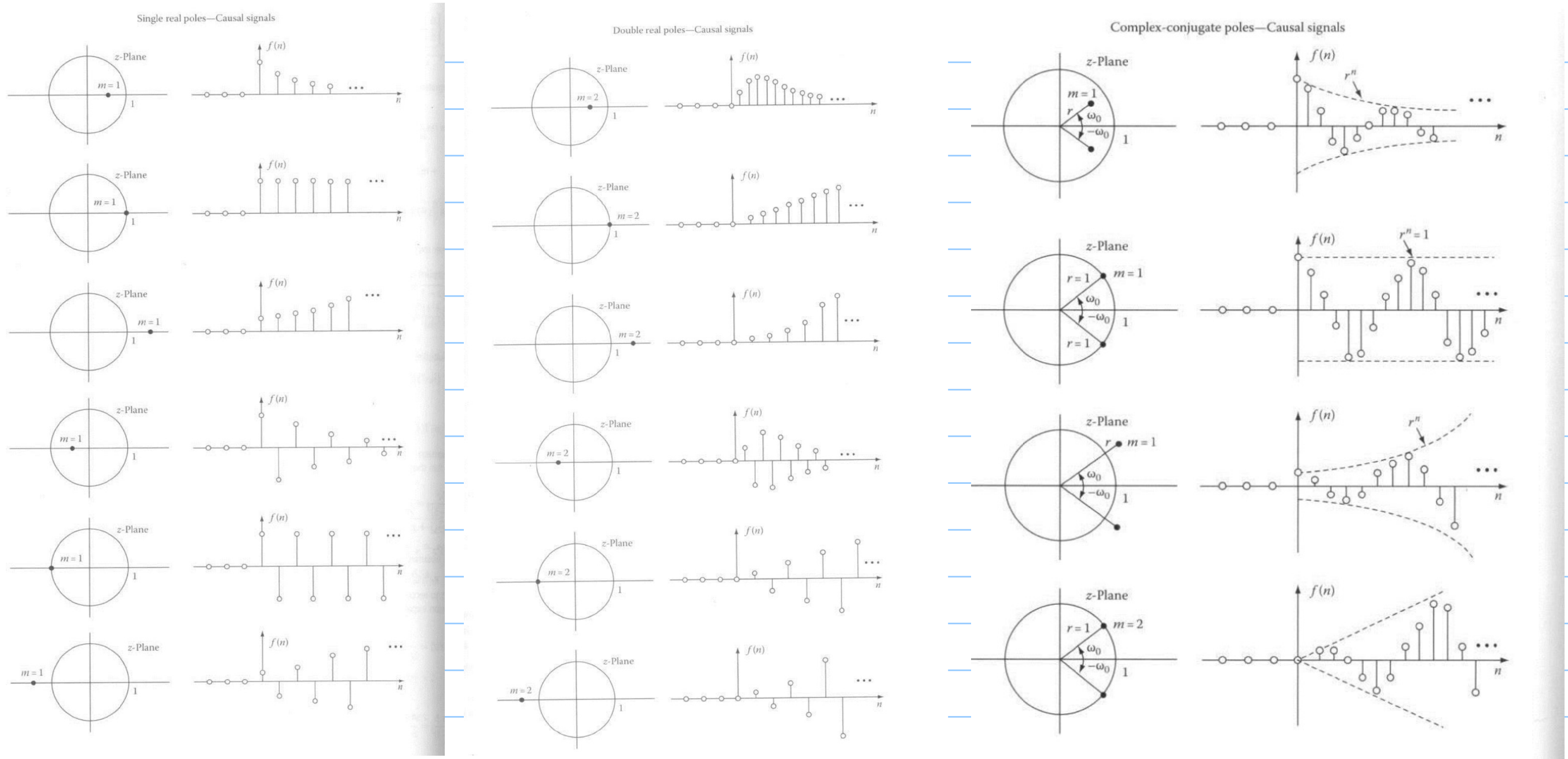
$$A(1) = 1 + a_1 + a_2 = (1 - r_1)(1 - r_2) > 0$$

$$A(-1) = 1 - a_1 + a_2 = (1 + r_1)(1 + r_2) > 0$$

Thus, $\left.\begin{aligned} -a_1 &< 1 + a_2 \\ a_1 &< 1 + a_2 \end{aligned}\right\} \implies |a_1| < 1 + a_2$

Hence, once again, the conditions are satisfied iff the real-valued roots are inside the unit circle.

# Some Typical Impulse Responses




From "Transforms and Applications Handbook", Alexander Poularikas (Ed.), 3rd edition, CRC Press, 2010