DFT as the samples of the DTFT

Suppose we assume that $x[n]$ is zero outside $[0, N-1]$

Then,

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

If we sample $X(e^{j\omega})$ at N uniformly spaced points, i.e., at

$\omega_k = \frac{2\pi k}{N}$ for $k = 0, 1, 2, \ldots, N-1$, then

$$X(e^{j\omega})\Big|_{\omega=\omega_k} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N} n} = X[k]$$

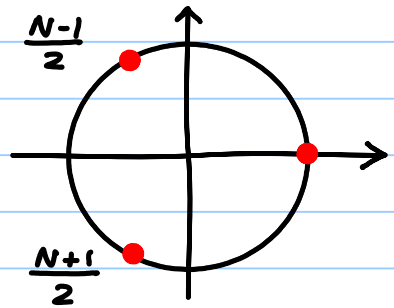
Thus, another interpretation of the DFT is viewing it as the samples of the DTFT.

Since $X(e^{j\omega_k}) = X[k]$, the inverse transform expression is identical, the implication of which is that we get $\tilde{x}[n]$ back, rather than $x[n]$. However, as before, $\tilde{x}[n] = x[n]$ for $0 \leq n \leq N-1$.

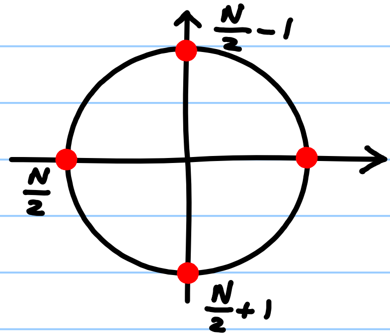
To interpret the periodicity, we see that $x[n]$ becomes $\tilde{x}[n]$ because of sampling $X(e^{j\omega})$. ["Sampling in one domain results in a periodic repetition in the other domain."]

If N is odd, there will not be a sample corresponding to $\omega = \pi$

If N is even, there will be a sample corresponding to $\omega = \pi$



$N = 3$



$N = 4$

Since both $x[n]$ and $X[k]$ are periodic, we need to consider the indices over the range $[0, N-1]$ only.

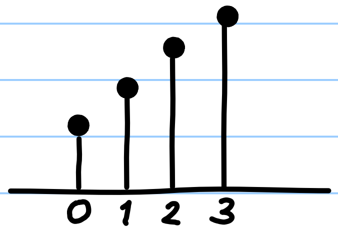
That is, the index 'l' is replaced by 'l mod N', and denoted by $\langle l \rangle_N \equiv l \bmod N$.

Properties

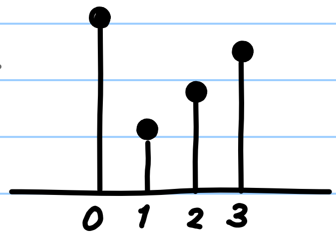
1) $a_1 x_1[n] + a_2 x_2[n] \longleftrightarrow a_1 X_1[k] + a_2 X_2[k]$

2) $x[n-n_0] \equiv x[\langle n-n_0 \rangle_N] \longleftrightarrow e^{-j\frac{2\pi k n_0}{N}} X[k]$

$x[\langle n-n_0 \rangle_N]$ is called as a circular shift.



$\xrightarrow[\text{by 1}]{\text{right shift}}$



same as left shift by 3

Note that there is no relationship, in general, between the DTFT of $x[n]$ and $x[\langle n-n_0\rangle_N]$. However, the corresponding DFTs share the relationship given above.

Also, since the sequence has the implied periodicity, a shift of $n-n_0$, where $0 \leq n_0 \leq N-1$, is the same as $n+m_0$ where $m_0 = N-n_0$.

3) $$e^{j\frac{2\pi \ell n}{N}} x[n] \longleftrightarrow X[k-\ell] \equiv X[\langle k-\ell\rangle_N]$$

4) $x[n] \circledast_N y[n] = \sum_{m=0}^{N-1} x[m]\, y[n-m] \longleftrightarrow X[k]\, Y[k]$

5) $x[n]\, y[n] \longleftrightarrow \frac{1}{N} X[k] \circledast_N Y[k]$

6) $\underline{x} = (x[0], x[1], \ldots, x[N-1])^T$

$$\|\underline{x}\|_2^2 = \underline{x}^H \underline{x}$$

$$\|\underline{X}\|_2^2 = \underline{X}^H \underline{X} = (\underline{W}\,\underline{x})^H (\underline{W}\,\underline{x})$$

$$= \underline{x}^H \underline{W}^H \underline{W}\, \underline{x}$$

$$= N\, \underline{x}^H \underline{x}$$

Hence, $\underline{x}^H \underline{x} = \frac{1}{N} \underline{X}^H \underline{X}$, i.e, $\|\underline{x}\|_2^2 = \frac{1}{N} \|\underline{X}\|_2^2$

7) $x^*[n] \longleftrightarrow X^*[-k] = X^*[N-k]$

$\Rightarrow$ $X[1] = X^*[N-1]$,
$X[2] = X^*[N-2]$, and so on.

Recovering the DTFT from the DFT

If the DTFT of an N-point sequence is sampled at a set of at least N uniformly spaced points, we can recover the DTFT from the DFT without any loss in information.

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi k n}{N}} \right] e^{-j\omega n}$$

$$= \sum_{k=0}^{N-1} X[k]\, P\left(\omega - \frac{2\pi k}{N}\right)$$

where $P(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1}{N} \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = e^{-j\omega(N-1)/2} \frac{\sin N\omega/2}{\sin \omega/2}$

Note that $P\left(\frac{2\pi k}{N}\right) = \begin{cases} 1 & k = 0 \\ 0 & k = 1, 2, \ldots, N-1 \end{cases}$