# EE5330 Oct. 29, 2013

Note Title 29-10-2013

A frequency of $F_o$ Hz gets mapped to $f = \frac{F_o}{F_s}$ in the DTFT domain.

This leads to the following:

$x_c(t) = \cos 2\pi F_o t$

$F_o = 8$ kHz, $F_s = 24$ kHz

$x[n] = \cos 2\pi \frac{8\times10^3}{24\times10^3} n$

$= \cos \frac{2\pi n}{3}$

$y_c(t) = \cos 2\pi F_o t$

$F_o = 16$ kHz, $F_s = 48$ kHz

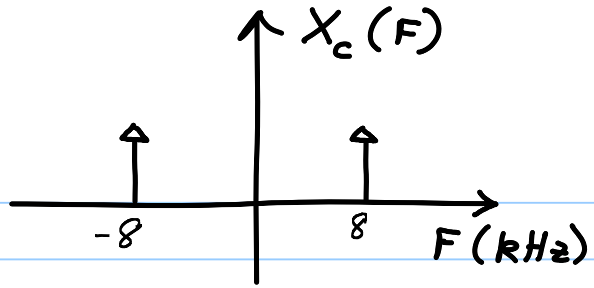
$y[n] = \cos 2\pi \frac{16\times10^3}{48\times10^3} n$

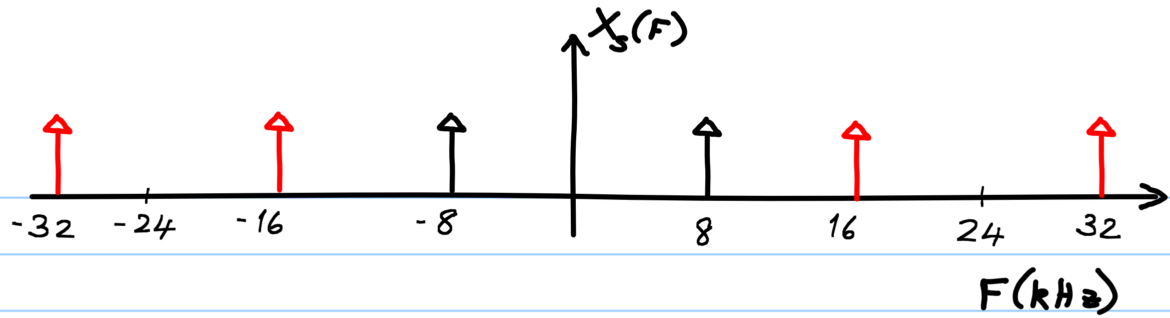
$= \cos \frac{2\pi n}{3}$

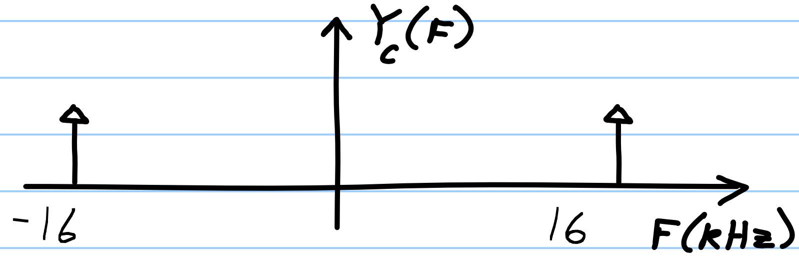
$= x[n]$

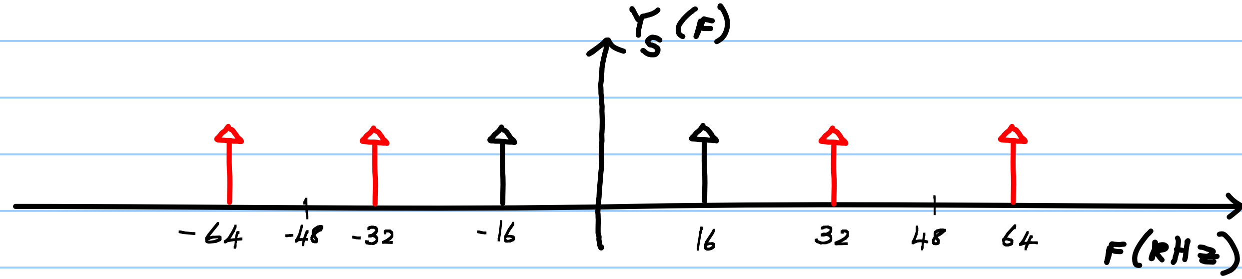
Thus, given $x[n] = \cos \frac{2\pi n}{3}$, one cannot tell whether it is a result of sampling an 8 kHz signal at 24 kHz or a 16 kHz signal at 48 kHz. To deduce the true signal frequency in Hz from the given sampled sequence, we need information about $F_s$

Note that $X_s(F)$ and $Y_s(F)$ have no ambiguity in revealing the true signal frequency.

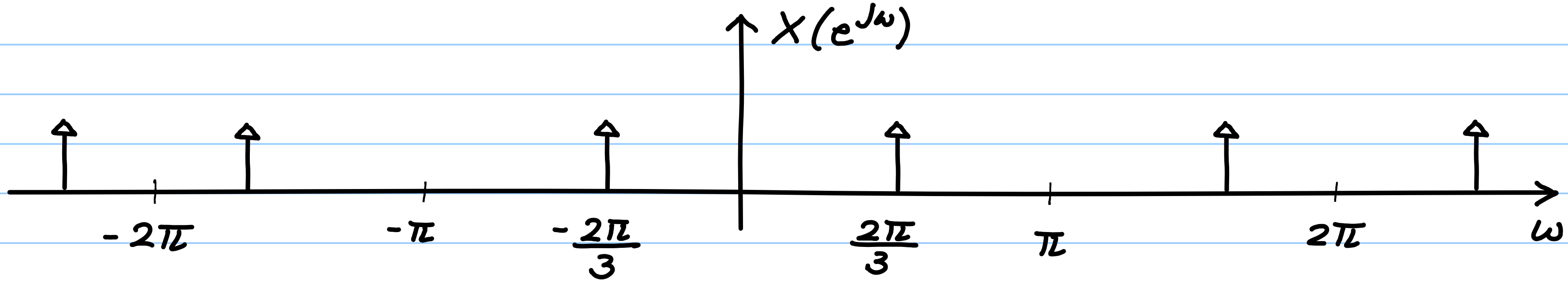



Both $X_s(F)$ and $Y_s(F)$ map to the same $X(e^{j\omega})$:



# The Discrete Fourier Transform (DFT)

Recall the various Fourier representations we have seen so far:

| Indep. Variable | Periodic ? | Spectrum | Periodic ? | |
|---|---|---|---|---|
| continuous | yes | line | no | CTFS |
| continuous | no | continuous | no | CTFT |
| discrete | no | continuous | yes | DTFT |
| discrete | yes | line | yes | DTFS |

$$x(t+T) = x(t) \qquad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \qquad a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\Omega_0 t} dt$$

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} dt$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$x[n+N] = x[n] \qquad x[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi k n}{N}} \qquad a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}$$

Suppose $x[n]$ is known for $n = 0, 1, 2, \ldots, N-1$. We define the DFT as follows:

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}}}$$

From the definition it follows that $X[k+N] = X[k]$ and hence the range of 'k' of interest is $\boxed{k = 0, 1, 2, \ldots, N-1}$.

The DFT can be expressed using matrix-vector notation.

$$\begin{bmatrix} & \uparrow & \\ \longleftarrow & e^{-j\frac{2\pi k n}{N}} & \longrightarrow n \\ & \downarrow k & \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\underset{N\times N}{\underline{W}} \qquad \underset{N\times 1}{\underline{x}} \qquad \underset{N\times 1}{\underline{X}}$$

It can easily be verified that $\underline{W}$ is full rank, i.e., rank $N$, and hence invertible. Therefore $\underline{x}$ can be obtained from $\underline{X}$ as follows:

$$\underline{x} = \underline{W}^{-1}\underline{X}$$

In equation form, the above can be expressed as,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi k n}{N}}$$

The inverse transform implies $x[n] = x[n+N]$. That is, even though no assumption was made about $x[n]$ outside $[0, N-1]$, the DFT framework imposes periodicity on $x[n]$.

Thus, both $x[n]$ and $X[k]$ are periodic. This is reminiscent of DTFS. In fact the DFT is nothing but a slightly modified version of the DTFS!

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi n k}{N}}$$

Thus,

$$\boxed{X[k] = N a_k}$$

Exercise

Show that $\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi k n}{N}}$ gives back $x[n]$.